

CONJUGACY PROBLEM IN GROUPS OF NON-ORIENTABLE 3-MANIFOLDS

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ABSTRACT. We prove that fundamental groups of non-orientable 3-manifolds have a solvable conjugacy problem, and construct an algorithm. Together with our earlier work on the conjugacy problem in groups on orientable geometrizable 3-manifolds, all π_1 of (geometrizable) 3-manifolds have a solvable conjugacy problem. As corollaries, both the twisted conjugacy problem in closed surface groups and the conjugacy problem in closed surface-by-cyclic groups, are solvable.

INTRODUCTION

Since formulated by M. Dehn in the early 1910's the word problem and conjugacy problem in finitely presented groups have become fundamental problems in combinatorial group theory. Following the work of Novikov [No] and further authors on their general unsolvability, it has become fairly natural to ask for any finitely presented group whether it admits a solution or not. For example in [De1, De2, De3], Dehn has solved those problems for fundamental groups of closed surfaces; his motivation was topological. Given a finite presentation of a group G , a solution to the *word problem* is an algorithm which given two elements $\omega, \omega' \in G$ as words in the generators and their inverses, decides whether $\omega = \omega'$ in G or not. A solution to the *conjugacy problem* is an algorithm which given $\omega, \omega' \in G$ decides whether $\exists h \in G$ such that $\omega' = h\omega h^{-1}$ in G or not. It turns out that existence of solutions does not depend of the finite presentation involved but only on the isomorphism class of G . We say that G has a *solvable word problem* (resp. *conjugacy problem*) if G admits a solution to the word problem (resp. conjugacy problem).

By a 3-manifold we mean a connected compact manifold of dimension 3 with boundary; a 3-manifold may be orientable or not. We work in the pl category; according to the *hauptvermutung* and Moise's theorem this is not restrictive. Following the work of Thurston (cf. [Th]) an oriented 3-manifold \mathcal{M} is *geometrizable* if the pieces obtained in

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its canonical topological decomposition (roughly speaking along essential spheres, discs and tori) have interiors which admit complete locally homogeneous riemannian metrics. It's an important result that every orientable 3-manifold is geometrizable; the work of Perelman following Hamilton program in the early 2000's, awarded by a Fields Medal in 2006, together with clarifications by numerous authors (cf. [BBBMP]) gives a proof of this statement. We make implicit the assumption that all non-orientable 3-manifolds considered have an orientation cover with total space a geometrizable orientable 3-manifold, by talking of geometrizable non-orientable 3-manifolds (this is a weak version of geometrization conjecture for non-orientable 3-manifolds, see [Sc] p.484–485). The reader who might not feel comfortable with Perelman's proof may consider this assumption as a restrictive hypothesis on the non-orientable 3-manifolds to which our method applies.

In the class of fundamental groups of geometrizable 3-manifolds, the word problem is known to be solvable, following early work of Waldhausen ([Wa]) as well as later work of Epstein and Thurston on automatic group theory (cf. [CEHLPT]). We have proved in [Pr] that groups of orientable geometrizable 3-manifolds have a solvable conjugacy problem; we will make use of those two results in our proof. We now focus on non-orientable 3-manifolds and construct an algorithm solving the conjugacy problem in their fundamental groups. In conclusion all 3-manifolds have fundamental groups with solvable conjugacy problem, which contrasts with higher dimensions. We also state as corollaries that the conjugacy problem is solvable in surface-by-cyclic groups and that the twisted conjugacy problem is solvable in surface groups (cf. §1).

The solution to the conjugacy problem in groups of non-orientable 3-manifolds does not follow as a direct consequence of the existence of a solution in the oriented case, since D.Collins and C.Miller have shown that the conjugacy problem can be unsolvable in a group even when solvable in an index 2 subgroup ([CM]). Nevertheless our strategy will consist essentially in reducing to the conjugacy problem in the oriented case.

We briefly emphasize two points which sound noteworthy to us. On the one hand the core of the algorithm makes use of basic solutions to the word and conjugacy problem in groups of orientable 3-manifolds, themselves reducing to basic solutions, built from biautomatic group theory, in groups of the basic pieces, Seifert fiber spaces and finite volume hyperbolic manifolds, and requires no naive enumerating algorithm. So one may expect to construct an efficient algorithm, provided that efficient process solving Dehn problems in groups of the basic pieces were known. On the other hand our general strategy to reduce the problem to similar problems in the orientation cover may help solving the conjugacy problem of a group G containing an index 2 subgroup H with solvable conjugacy problem. One of our key ingredients here is that centralizers in G of non trivial elements of H are

fair enough: they can be computed and have solvable Dehn problems. The general translation which would ensure a solution in an abstract group G containing an index 2 subgroup H would become that (i) one can decide whether two arbitrary elements of order 2 are conjugate in G , and (ii) for any $g \in G \setminus H$, of order $\neq 2$, $Z_G(g^2)$ can be computed and has a solvable conjugacy problem.

1. STATEMENT OF THE RESULTS

This work is mainly devoted to prove the following result:

Theorem A. [Main result.] *Fundamental groups of non-orientable geometrizable 3-manifolds have solvable conjugacy problem.*

Together with a solution in the oriented case (cf. [Pr]), one obtains:

Theorem B. *Fundamental groups of geometrizable 3-manifolds have solvable conjugacy problem. Topologically rephrased, given any pair of loops γ, γ' in a geometrizable 3-manifold, one can decide whether γ, γ' are freely homotopic.*

Note that Theorem A does not follow as an easy consequence of the oriented case; indeed the usual technique which consists in translating the problem to the oriented cover fails to give an immediate answer, for the conjugacy problem can be unsolvable in a group G even when solvable in an index 2 subgroup H (cf. [CM]). Note also that we do not only show existence of a solution, but rather give a constructive process to build an algorithm solving the problem. Moreover whenever u, v are conjugate the algorithm implicitly produces a conjugating element h such that $u = hvh^{-1}$.

Theorem B has several consequences. Concerning decision problems relative to boundary subgroups in groups of geometrizable 3-manifolds one obtains:

Theorem C. *Let \mathcal{M} be a geometrizable 3-manifold and $\mathcal{F} \subset \partial\mathcal{M}$ a compact connected surface. Denote $G = \pi_1(\mathcal{M})$ and $H = i_*(\pi_1(\mathcal{F}))$; there exists algorithms which decide for any $g \in G$ respectively whether $g \in H$ and whether g is conjugate to an element of H . Topologically rephrased given any loop γ (resp. $*$ -based loop, $*$ $\in \mathcal{F}$) in \mathcal{M} one can decide whether up to homotopy (resp. $*$ -fixed homotopy) γ lies in \mathcal{F} .*

Proof that Theorem B \implies Theorem C. Double the 3-manifold \mathcal{M} along the identity on \mathcal{F} to obtain $\mathcal{M} \sqcup_{\mathcal{F}} \mathcal{M}$. The proof of lemma 1.2 of [Pr], as well as the observation that the orientation cover of $\mathcal{M} \sqcup_{\mathcal{F}} \mathcal{M}$ is the double of the orientation cover of \mathcal{M} along the lift(s) of \mathcal{F} show that $\mathcal{M} \sqcup_{\mathcal{F}} \mathcal{M}$ is geometrizable. Its group splits into an amalgam of two copies of $G = \pi_1(\mathcal{M})$ along the identity of $K = \pi_1(\mathcal{F})$, say $\Gamma = G *_K G$. Given g lying in the G left factor we denote \bar{g} the corresponding element of the G right factor. Since the gluing map is the

identity $h = \bar{h}$, one obtains by applying elementary facts on amalgams (cf. Corollary 4.4.2 and Theorem 4.6 in [MKS]) that g, \bar{g} are equal (resp. conjugate) in Γ if and only if $g \in K$ (resp. g is conjugate in G to some $h \in K$). Hence with a solution to the word problem (resp. conjugacy problem) in Γ provided by Theorem B, it suffices to decide whether $g = \bar{g}$ (respectively g and \bar{g} are conjugate) or not. \square

A second consequence concerns the conjugacy problem in surface by cyclic groups. It has been proved in [BMMV] that (f.g. free)-by-cyclic groups have solvable conjugacy problem. As a complementary result we can deduce from theorem B together with the Dehn-Nielsen theorem the same statement concerning (closed surface)-by-cyclic groups, so that any (compact surface)-by-cyclic group turns out to have a solvable conjugacy problem.

Theorem D. *Closed surface-by-cyclic groups have solvable conjugacy problem.*

Proof that Theorem B \implies Theorem D. Let \mathcal{F} be a closed surface, $K = \pi_1(\mathcal{F})$, and G be an extension of K by a cyclic group C . In case C is finite, G is biautomatic and has solvable conjugacy problem (cf. [CEHLPT]). In case C is infinite, the extension splits as $G = K \rtimes_{\phi} \mathbb{Z}$ for some $\phi \in \text{Aut}(K)$. The Dehn-Nielsen theorem (Theorem 3.4.6, [CGKZ]) shows that ϕ is induced by a homeomorphism f of the surface \mathcal{F} so that G is isomorphic to the fundamental group of the bundle over \mathbf{S}^1 with fiber \mathcal{F} and sewing map f . It follows from the Thurston geometrization theorem ([Th]) that such a bundle is geometrizable so that Theorem B shows that G has solvable conjugacy problem. \square

A third consequence concerns the twisted conjugacy problem in surface groups. Given a group G and an automorphism ϕ of G the twisted conjugacy problem is solvable in (G, ϕ) if one can algorithmically decide given any couple $u, v \in G$ whether there exists $g \in G$ such that $\phi(g)ug^{-1} = v$. The twisted conjugacy problem is solvable in G if it is solvable in (G, ϕ) for any automorphism $\phi \in \text{Aut}(G)$.

Theorem E. *Closed surface groups have solvable twisted conjugacy problem.*

Together with the case of f.g. free groups (cf. [BMMV]) the same result holds for (compact surface) groups.

Proof that Theorem D \implies Theorem E. Let $K = \pi_1(\mathcal{F})$ for a closed surface \mathcal{F} , ϕ an automorphism of K and $G = K \rtimes_{\phi} \mathbb{Z} = \langle G, t \mid tgt^{-1} = \phi(g) \rangle$. Given $u, v \in K$, there exists $g \in K$ such that $\phi(g)ug^{-1} = v$ if and only if g conjugates $t^{-1}u$ into $t^{-1}v$ in G . Existence of such $g \in K$ is equivalent to $t^{-1}u$ conjugate to $t^{-1}v$ in G . For if there exists ht^p with $h \in K$, $n \in \mathbb{Z}$, which conjugates $t^{-1}u$ into $t^{-1}v$, so does $h_n = ht^p(t^{-1}u)^n$ for any $n \in \mathbb{Z}$, and in particular for h_p which belongs to K .

Hence a solution to the conjugacy problem in G provides a solution to the twisted conjugacy problem in (K, ϕ) . \square

2. PROOF OF THEOREM A

We are now devoted to the proof of the main result enounced in §1 upon conjugacy problem in groups of non-orientable 3-manifolds. Note that we not only prove the existence of a solution: we rather show how, given a triangulation of \mathcal{M} , one can construct the algorithm.

We start from a non-orientable (geometrizable) 3-manifold \mathcal{M} , given by a triangulation, and we construct an algorithm which solves conjugacy problem in $\pi_1(\mathcal{M})$. The process is done in four steps. In the first step we reduce to the closed irreducible case, *i.e.* we prove that from solutions in groups of closed irreducible non-orientable 3-manifolds one recovers solutions in groups of all non-orientable 3-manifolds. In the second step we construct the orientation cover $p : \mathcal{N} \longrightarrow \mathcal{M}$ of \mathcal{M} , cover involution $\sigma : \mathcal{N} \longrightarrow \mathcal{N}$ and complete topological decompositions of \mathcal{M}, \mathcal{N} ; this is done by applying known algorithms for the decomposition of orientable 3-manifolds to \mathcal{N} (using Haken normal surfaces theory, [JT, JLR]) then deforming the surfaces obtained in \mathcal{N} so that they become almost σ -invariant. In the third step we construct the induced splittings \mathbf{M}, \mathbf{N} as graphs of groups of $\pi_1(\mathcal{M}) = \pi_1(\mathbf{M})$, $\pi_1(\mathcal{N}) = \pi_1(\mathbf{N})$ and covering $\mathbf{p} : \mathbf{N} \longrightarrow \mathbf{M}$ of graphs of groups which makes natural dealing with elements in $\pi_1(\mathbf{M}), \pi_1(\mathbf{N})$; we also state all the basic algorithms needed in the fourth and final step, where is given the core of the algorithm.

2.1. Step 1: Reducing to the closed irreducible case. The following preliminary step reduces the proof to the case of closed irreducible geometrizable 3-manifolds.

Lemma 2.1. *Conjugacy problem in groups of non-orientable geometrizable 3-manifolds reduces to conjugacy problem in groups of closed irreducible geometrizable 3-manifolds. Moreover given a triangulation of \mathcal{M} the reduction process can be achieved in a constructive way.*

Proof. Let \mathcal{M} be a non-orientable geometrizable 3-manifold; we are concerned with the conjugacy problem in $\pi_1(\mathcal{M})$. We process the reduction in two steps, first reducing to closed manifolds and then to closed irreducible ones.

Gluing a 3-ball to each spherical component of $\partial\mathcal{M}$ leaves $\pi_1(\mathcal{M})$ unchanged; so we suppose in the following that \mathcal{M} has no spherical boundary component. If $\partial\mathcal{M}$ is non-empty, double the manifold \mathcal{M} along its boundary to obtain the closed non-orientable 3-manifold that we shall denote $2\mathcal{M}$. Lemma 1.1 of [Pr] asserts that the inclusion map of \mathcal{M} in $2\mathcal{M}$ induces an embedding of $\pi_1(\mathcal{M})$ in $\pi_1(2\mathcal{M})$, and that $u, v \in \pi_1(\mathcal{M})$ are conjugate in $\pi_1(\mathcal{M})$ if and only if they are conjugate in $\pi_1(2\mathcal{M})$; hence the conjugacy problem in $\pi_1(\mathcal{M})$ reduces to the conjugacy problem in $\pi_1(2\mathcal{M})$. Moreover the closed manifold

$2\mathcal{M}$ is geometrizable. Indeed if $\overline{\mathcal{M}}$ and $\overline{2\mathcal{M}}$ denote respectively the orientation covers of \mathcal{M} and $2\mathcal{M}$, one has that $\overline{2\mathcal{M}}$ is the double of $\overline{\mathcal{M}}$. Since \mathcal{M} is geometrizable so is $\overline{\mathcal{M}}$ and lemma 1.2 of [Pr] states that $\overline{2\mathcal{M}}$, and consequently $2\mathcal{M}$ are geometrizable. Hence the conjugacy problem in $\pi_1(\mathcal{M})$ reduces to conjugacy problem in the group of the closed geometrizable 3-manifold $2\mathcal{M}$. Moreover if \mathcal{M} is given by a triangulation, the reduction can be achieved in a constructive way for one can constructively produce a triangulation of $2\mathcal{M}$ and the natural embedding from $\pi_1(\mathcal{M})$ to $\pi_1(2\mathcal{M})$.

We are now concerned with the second step, and suppose \mathcal{M} to be moreover closed. A Kneser-Milnor decomposition splits \mathcal{M} in a connected sum of the prime closed geometrizable (perhaps orientable) factors $\mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_i$ and $\pi_1(\mathcal{M})$ splits as the free product of the groups $\pi_1(\mathcal{M}_1), \pi_1(\mathcal{M}_2), \dots, \pi_1(\mathcal{M}_n)$. Basic result upon conjugacy in free products (cf. theorem 4.2, §4.1, [MKS]) shows that the conjugacy problem in $\pi_1(\mathcal{M})$ reduces to conjugacy problems in each of the $\pi_1(\mathcal{M}_i)$. Now either \mathcal{M}_i is an \mathbf{S}^2 -bundle over \mathbf{S}^1 , and in that case $\pi_1(\mathcal{M}_i) \simeq \mathbb{Z}$, or \mathcal{M}_i is irreducible. Hence conjugacy problem in $\pi_1(\mathcal{M})$ reduces to conjugacy problem in groups of closed irreducible geometrizable 3-manifolds. If \mathcal{M} is given by a triangulation an algorithm given in [JT] for a Kneser-Milnor decomposition allows to process the reduction in a constructive way. \square

According to lemma 2.1, in order to achieve proof of Theorem A we are left with the case of groups of closed irreducible geometrizable non-orientable 3-manifolds.

- In the following \mathcal{M} stands for a closed irreducible geometrizable non-orientable 3-manifold, and $p : \mathcal{N} \rightarrow \mathcal{M}$ for the orientation cover of \mathcal{M} .

2.2. Step 2: Algorithms for the topological decomposition of \mathcal{M} and \mathcal{N} . We start from a closed irreducible non-orientable geometrizable 3-manifold \mathcal{M} given by a triangulation. We show how to algorithmically construct the orientation cover $p : \mathcal{N} \rightarrow \mathcal{M}$ and fair enough topological decompositions of \mathcal{M}, \mathcal{N} .

Lemma 2.2 (Algorithm $\mathfrak{T}op1$ for the orientation cover). *Given a triangulation of \mathcal{M} one can algorithmically produce a triangulation of its orientation cover total space \mathcal{N} as well as the covering map $p : \mathcal{N} \rightarrow \mathcal{M}$ and covering automorphism $\sigma : \mathcal{N} \rightarrow \mathcal{N}$.*

Proof. The triangulation of \mathcal{M} can be easily given as a triangulation of a pl-ball \mathcal{B} together with a gluing of pairs of triangles in $\partial\mathcal{B}$. Pick an orientation of \mathcal{B} ; it induces an orientation of each triangle in $\partial\mathcal{B}$. Identify paired triangles in $\partial\mathcal{B}$ each time their gluing preserves orientation, to obtain a new oriented pl-manifold \mathcal{C} together with orientation reversing gluing of pairs of triangles in $\partial\mathcal{C}$. Consider a copy \mathcal{C}' of \mathcal{C} and give \mathcal{C}' the reverse orientation. For each triangle δ in $\partial\mathcal{C}$ denote

by δ' its copy in $\partial\mathcal{C}'$. For each gluing of triangles δ_1, δ_2 in $\partial\mathcal{C}$, glue coherently in $\mathcal{C} \cup \mathcal{C}'$, δ_1 with δ'_2 and δ'_1 with δ_2 (cf. figure 1). The manifold obtained is \mathcal{N} together with a triangulation, and the construction implicitly produces the covering map $p : \mathcal{N} \rightarrow \mathcal{M}$ as well as the covering automorphism. \square

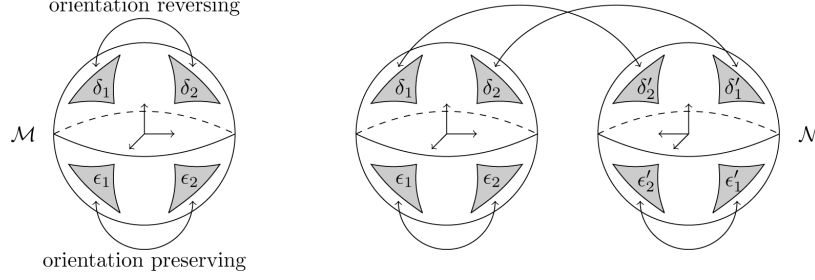


FIGURE 1. Construction of \mathcal{N} from \mathcal{M} given as a pl-ball with identifications on the boundary.

- Apply algorithm **Top1** to construct a triangulation of the 2-cover \mathcal{N} of \mathcal{M} as well as the covering projection p and involution σ .

The oriented manifold \mathcal{N} may be reducible. In such a case (cf. [Sw]) \mathcal{N} contains a compact surface Σ whose components are σ -invariant essential spheres such that:

- \mathcal{N} cut along Σ decomposes into σ -invariant components: $\mathcal{N}_1, \mathcal{N}_2, \dots, \mathcal{N}_p$ and each manifold \mathcal{N}_i obtained by gluing a ball to each $\mathbf{S}^2 \subset \partial\mathcal{N}_i$ is irreducible and not simply connected;
- let n be the number of non-separating components in Σ ; $\pi_1(\mathcal{N})$ decomposes as a free product of $\pi_1(\mathcal{N}_1), \dots, \pi_1(\mathcal{N}_p)$ and of a free group with rank n : $\pi_1(\mathcal{N}) \simeq \pi_1(\mathcal{N}_1) * \dots * \pi_1(\mathcal{N}_p) * F_n$;

and Σ has image in \mathcal{M} a compact surface $\Pi = p(\Sigma)$ whose components are two-sided projective planes such that:

- \mathcal{M} cut along Π has components $\mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_p$ and the covering projection sends \mathcal{N}_i onto \mathcal{M}_i ;
- it induces a splitting of $\pi_1(\mathcal{M})$ as a graph of groups whose vertex groups are $\pi_1(\mathcal{M}_1), \pi_1(\mathcal{M}_2), \dots, \pi_1(\mathcal{M}_p)$ and edge groups all have order 2.

Lemma 2.3 (Algorithm **Top2** for coherent decompositions along \mathbf{S}^2 of \mathcal{N} and \mathbf{P}^2 of \mathcal{M}). *One can algorithmically find systems of pairwise disjoint essential σ -invariant spheres Σ in \mathcal{N} and projective planes Π in \mathcal{M} , as above.*

Proof. Apply algorithm 7.1 of [JT] (or, for a bound on complexity, an improved algorithm in [JLR]), to find, if any, an essential sphere \mathcal{S}_0 in \mathcal{N} . If none exists then $\Sigma = \Pi = \emptyset$ and the process stops. Otherwise apply to \mathcal{S}_0 the following classical argument (cf. [To]) to construct

a σ -invariant essential sphere \mathcal{S} in \mathcal{N} . Since \mathcal{M} is irreducible then $\sigma\mathcal{S}_0 \cap \mathcal{S}_0 \neq \emptyset$ for otherwise $p(\mathcal{S}_0)$ would be an essential sphere in \mathcal{M} . If $\sigma\mathcal{S}_0 = \mathcal{S}_0$ there is nothing left to prove so suppose it does not occur. Deform slightly \mathcal{S}_0 so that $\sigma\mathcal{S}_0 \cap \mathcal{S}_0$ consists in $n > 0$ closed simple curves. Consider such a curve, which in addition bounds an innermost disk \mathcal{D} in $\sigma\mathcal{S}_0$ (i.e. a disk in $\sigma\mathcal{S}_0 \setminus (\sigma\mathcal{S}_0 \cap \mathcal{S}_0)$), as well as a disk \mathcal{D}' in \mathcal{S}_0 . Consider the two spheres $\mathcal{S}_1 = \mathcal{S}_0 \cup \mathcal{D} \setminus \text{int}(\mathcal{D}')$ and $\mathcal{S}_2 = \mathcal{D} \cup \mathcal{D}'$ and perform small isotopies (see figure 2) so that $\mathcal{S}_i = \sigma\mathcal{S}_i$ or $\mathcal{S}_i \cap \sigma\mathcal{S}_i$ has fewer than n components ($i = 1, 2$).

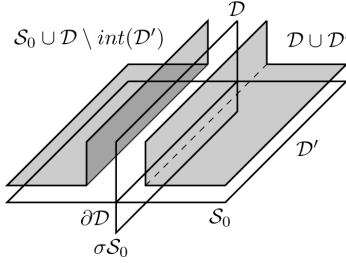


FIGURE 2. By considering small enough collar neighborhood $N(\mathcal{S}_0)$ of \mathcal{S}_0 and $\sigma N(\mathcal{S}_0)$ of $\sigma\mathcal{S}_0$ in a subdivision of the triangulation of \mathcal{N} , one can deform by isotopy the spheres $\mathcal{S}_1 = \mathcal{S}_0 \cup \mathcal{D} \setminus \text{int}(\mathcal{D}')$ and $\mathcal{S}_2 = \mathcal{D} \cup \mathcal{D}'$ such that either $\sigma(\mathcal{S}_i) = \mathcal{S}_i$ or they become transverse and $\mathcal{S}_i \cap \sigma\mathcal{S}_i$ has fewer components, $i = 1, 2$.

Moreover at least one of $\mathcal{S}_1, \mathcal{S}_2$ does not bound a ball in \mathcal{N} : suppose on the contrary that $\mathcal{S}_1, \mathcal{S}_2$ bound respective balls $\mathcal{B}_1, \mathcal{B}_2$ then either: (i) $\mathcal{B}_1 \supset \mathcal{S}_2$ and \mathcal{S}_0 also bounds a ball included in \mathcal{B}_1 , or (ii) $\mathcal{B}_1 \cap \mathcal{B}_2 = \emptyset$ and there exists a collar neighborhood $N(\mathcal{D})$ of \mathcal{D} such that $\mathcal{B}_1 \cup \mathcal{B}_2 \cup N(\mathcal{D})$ is a ball bounded by \mathcal{S}_0 ; it would contradict that \mathcal{S}_0 is essential. Apply the \mathbf{S}^3 recognition algorithm (cf. [Ru]) to each component of $\mathcal{N} \setminus \mathcal{S}_1, \mathcal{N} \setminus \mathcal{S}_2$ with balls glued on their boundary, to check which of $\mathcal{S}_1, \mathcal{S}_2$, say \mathcal{S}_1 , does not bound a ball. Then apply the same process to \mathcal{S}_1 instead of \mathcal{S}_0 , and so on. Since the number of components of $\sigma\mathcal{S}_i \cap \mathcal{S}_i$ decreases it will finally stop, leading us with a σ -invariant essential sphere \mathcal{S} in \mathcal{N} .

Cut \mathcal{N} along \mathcal{S} and then glue balls $\mathcal{B}_1, \mathcal{B}_2$ to its boundary to obtain \mathcal{N}_1 and (possibly) \mathcal{N}_2 . Since σ preserves \mathcal{S} and reverses the orientation both on \mathcal{N} and \mathcal{S} it necessarily preserves each component of $\mathcal{N} \setminus \mathcal{S}$. Restrict then extend σ to an involution of \mathcal{N}_i with fixed points ($i = 1, 2$). Then apply the same argument as above to search for essential σ -invariant spheres in \mathcal{N}_1 and \mathcal{N}_2 . We furthermore need to deform each such sphere so that it lies in $\mathcal{N} \setminus \mathcal{S}$, that is, so that it does not intersect neither \mathcal{B}_1 nor \mathcal{B}_2 .

Suppose without loss of generality that we have found an essential σ -invariant sphere \mathcal{S}_0 in \mathcal{N}_1 which intersects \mathcal{B}_1 . After slightly deforming \mathcal{S}_0 by isotopy, $\mathcal{S}_0 \cap \partial\mathcal{B}_1$ consists in $m > 0$ simple closed curves. Let \mathcal{D}' be an innermost disk in \mathcal{S}_0 ; $\partial\mathcal{D}'$ bounds a disk \mathcal{D} in $\partial\mathcal{B}_1$. If \mathcal{D}' lies

in \mathcal{B}_1 , consider $\mathcal{S}_1 = \mathcal{S}_0 \cup (\mathcal{D} \cup \sigma\mathcal{D}) \setminus \text{int}(\mathcal{D}' \cup \sigma\mathcal{D}')$ and deform it by a small isotopy (cf. figure 3) so that \mathcal{S}_1 becomes a σ -invariant essential sphere such that $\mathcal{S}_1 \cap \partial\mathcal{B}_1$ has less than m components.

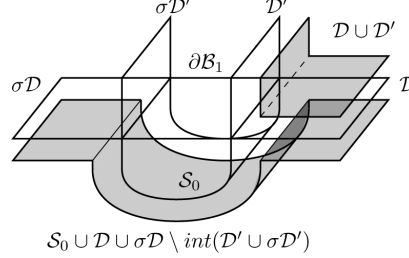


FIGURE 3. By considering small enough collar neighborhood $N(\mathcal{S}_0)$ of \mathcal{S}_0 and $N(\partial\mathcal{B}_1)$ of $\partial\mathcal{B}_1$ in a subdivision of the triangulation of \mathcal{N} , one can deform by isotopy the spheres $\mathcal{S}_1 = \mathcal{S}_0 \cup (\mathcal{D} \cup \sigma\mathcal{D}) \setminus \text{int}(\mathcal{D}' \cup \sigma\mathcal{D}')$ and $\mathcal{S}_2 = \mathcal{D} \cup \mathcal{D}'$ so that their number of intersections with $\partial\mathcal{B}_1$ decreases.

If \mathcal{D}' does not lie in \mathcal{B}_1 , consider the two spheres \mathcal{S}_1 as above and $\mathcal{S}_2 = \mathcal{D} \cup \mathcal{D}'$ (cf. figure 3). At least one of $\mathcal{S}_1, \mathcal{S}_2$ does not bound a ball (for the same reason as above); use the \mathbf{S}^3 recognition algorithm to check so. If \mathcal{S}_1 does not bound a ball, as above, after a small isotopy, \mathcal{S}_1 is a σ -invariant essential sphere such that $\mathcal{S}_1 \cap \partial\mathcal{B}_1$ has less than m components. If \mathcal{S}_2 does not bound a ball, after a small isotopy, one obtains an essential sphere \mathcal{S}_2 ; but \mathcal{S}_2 may be not σ -invariant. Since \mathcal{M} is irreducible one has $\mathcal{S}_2 \cap \sigma\mathcal{S}_2 \neq \emptyset$ and using the same procedure as above one constructs a σ -invariant essential sphere \mathcal{S}_3 , included in $\mathcal{S}_2 \cup \sigma\mathcal{S}_2$ up to small isotopy, such that $\mathcal{S}_3 \cap \partial\mathcal{B}_1$ has less than m components.

Applying this process while it's possible one finally obtains, if any, a σ -invariant essential sphere in \mathcal{N}_1 which does not intersect \mathcal{B}_1 nor \mathcal{B}_2 . Cut \mathcal{N}_1 along this sphere, glue balls and then apply the same process to the manifolds obtained, while they do contain an essential sphere. According to the Kneser-Milnor theorem it will finally stop, leading us with the compact surface Σ union of σ -invariant essential spheres in \mathcal{N} , whose image in \mathcal{M} gives the compact surface Π union of two-sided projective planes. \square

- Apply algorithm **T**op2 to find surface Σ in \mathcal{N} made of essential σ -invariant spheres and surface Π in \mathcal{M} made of two-sided projective planes.

Cut \mathcal{M} along Π to obtain the pieces $\mathcal{M}_1, \mathcal{M}_2, \dots, \mathcal{M}_p$ of \mathcal{M} , and \mathcal{N} along Σ to obtain the pieces $\mathcal{N}_1, \mathcal{N}_2, \dots, \mathcal{N}_p$ of \mathcal{N} ; the involution σ restricts on each of the \mathcal{N}_i to a free involution with quotient \mathcal{M}_i . Consider $\hat{\mathcal{N}}_1, \hat{\mathcal{N}}_2, \dots, \hat{\mathcal{N}}_p$ by filling the spheres in the boundary of the \mathcal{N}_i with balls. Each involution $\sigma : \mathcal{N}_i \rightarrow \mathcal{N}_i$ extends uniquely, up to isotopy, to an involution $\sigma : \hat{\mathcal{N}}_i \rightarrow \hat{\mathcal{N}}_i$ with quotient the orbifold $\hat{\mathcal{M}}_i$ obtained from \mathcal{M}_i by gluing cones over \mathbf{P}^2 on each projective plane

of $\partial\mathcal{M}_i$.

Each of the $\hat{\mathcal{N}}_i$, $i = 1 \dots p$ is irreducible. Since $\hat{\mathcal{N}}_i$ is geometrizable, there exists $\Omega_i \subset \hat{\mathcal{N}}_i$ such that either $\Omega_i = \emptyset$ or Ω_i is a two-sided compact surface whose components are essential tori which is minimal and unique up to isotopy, and each component of $\mathcal{N}_i \setminus \Omega_i$ is either a Seifert fiber space or a finite volume hyperbolic manifold; a so called *canonical family of tori*. If $\Omega_i \neq \emptyset$, consider such a surface Ω_i which moreover verifies:

- Ω_i lies in \mathcal{N}_i , and
- $p(\Omega_i)$ is a two-sided compact surface Ξ_i in \mathcal{M}_i whose components are essential tori and Klein bottles, and
- if $\hat{\mathcal{N}}_i$ is not a \mathbf{T}^2 bundle over \mathbf{S}^1 modeled on *Sol*: for each component $\mathcal{T} \subset \Omega_i$, if $\sigma\mathcal{T} \not\subset \Omega_i$ then \mathcal{T} and $\sigma\mathcal{T}$ cobound in $\hat{\mathcal{N}}_i$ a component homeomorphic to $\mathbf{T}^2 \times \mathbf{I}$ and preserved by σ .

Such surfaces Ω_i in \mathcal{N}_i and Ξ_i in \mathcal{M}_i are what we call *coherent JSJ decompositions* of $\mathcal{M}_i, \mathcal{N}_i$.

Lemma 2.4 (Algorithm $\mathfrak{T}\mathfrak{o}\mathfrak{p}3$ for the JSJ decompositions). *One can algorithmically construct coherent JSJ decompositions $\Omega_i \subset \mathcal{N}_i$ and $\Xi_i \subset \mathcal{M}_i$ ($i = 1 \dots p$).*

Proof. During the whole proof we note $\hat{\mathcal{N}}$, \mathcal{N} and \mathcal{M} rather than $\hat{\mathcal{N}}_i$, \mathcal{N}_i and \mathcal{M}_i . Apply the algorithm 8.2 of [JT] to find the JSJ decomposition Ω (as well as the characteristic Seifert submanifold) of $\hat{\mathcal{N}}$. Apply the same argument as in the proof of the last lemma to deform the tori in Ω so that they all lie in the interior of \mathcal{N} .

Deform slightly Ω so that $\Omega \cap \sigma\Omega$ is either empty or consists in simple closed curves and in some of the tori \mathcal{T}_j 's. Then deform Ω so that each closed curve component in $\Omega \cap \sigma\Omega$ becomes essential in Ω . For suppose that $\sigma\mathcal{T}_1 \cap \mathcal{T}_2$ has component a non essential closed curve, then it must contain a curve bounding an innermost disk \mathcal{D} in $\sigma\mathcal{T}_1$ such that $\partial\mathcal{D}$ bounds a disk \mathcal{D}' in \mathcal{T}_2 . Change in Ω , \mathcal{T}_1 into $\mathcal{T}_1 \cup \sigma\mathcal{D}' \setminus \text{int}(\sigma\mathcal{D})$ and \mathcal{T}_2 into $\mathcal{T}_2 \cup \mathcal{D} \setminus \text{int}(\mathcal{D}')$, so that after a small isotopy the number of components in $\Omega \cap \sigma\Omega$ decreases (cf. figure 4). Apply this process until each closed curve in $\Omega \cap \sigma\Omega$ becomes essential in Ω .

Let \mathcal{T}_j be a component of Ω such that $\sigma\mathcal{T}_j \cap \Omega$ consists in simple closed essential curves. We prove first that $\sigma\mathcal{T}_j \cap \Omega$ cannot consist in exactly one essential curve γ . For, let \mathcal{T} be a component of Ω such that $\sigma\mathcal{T}_j \cap \mathcal{T} = \gamma$; necessarily the torus \mathcal{T} is non-separating. Consider a regular neighborhood $V(\Omega)$ of Ω in \mathcal{N} and $\mathcal{N}' = \mathcal{N} \setminus \text{int}(V(\Omega))$; γ gives rise to two disjoint essential curves γ^-, γ^+ in two different components of $\partial\mathcal{N}'$, and γ, γ' cobound an essential annulus in a component \mathcal{N}'' of \mathcal{N}' . Necessarily \mathcal{N}'' is a Seifert fiber space, with at least two boundary components. If $\mathcal{N}'' \approx \mathbf{T}^2 \times \mathbf{I}$ then γ^-, γ^+ are regular fibers in a Seifert fibration of $\mathcal{N}'' = \mathcal{N}'$ which extends to \mathcal{N} . This leads to a

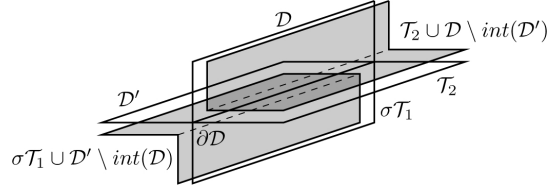


FIGURE 4. By considering small enough collar neighborhood $N(\mathcal{T}_2)$ of \mathcal{T}_2 and $N(\sigma\mathcal{T}_1)$ of $\sigma\mathcal{T}_1$ in a subdivision of the triangulation of \mathcal{N} , one can deform by isotopy the tori $\mathcal{T}_2 \cup \mathcal{D} \setminus \text{int}(\mathcal{D}')$ and $\sigma\mathcal{T}_1 \cup \mathcal{D}' \setminus \text{int}(\mathcal{D})$ so that their number of intersections decreases.

contradiction. If $\mathcal{N}'' \not\approx \mathbf{T}^2 \times \mathbf{I}$, with Lemma II.2.8 in [JS], γ^-, γ^+ are homotopic to regular fibers of a fibration of \mathcal{N}'' and the same argument shows that $\Omega \setminus \mathcal{T}$ is also a JSJ decomposition of \mathcal{N} , which contradicts the minimality of Ω . This leads to a contradiction.

Now let \mathcal{T}_j be a component of Ω such that $\sigma\mathcal{T}_j \cap \Omega$ consists in at least two simple closed essential curves. The family of essential curves is pairwise disjoint so that they cut $\sigma\mathcal{T}_j$ into annuli. Let γ, γ' be two such curves which cobound an innermost annulus \mathcal{A} (i.e. $\mathcal{A} \cap \Omega$ consists in γ, γ'). Let $\mathcal{T}, \mathcal{T}' \subset \Omega$ be such that $\gamma \subset \mathcal{T}$ and $\gamma' \subset \mathcal{T}'$. Suppose that $\mathcal{T} \neq \mathcal{T}'$; consider as above $\mathcal{N}' = \mathcal{N} \setminus \text{int}(V(\Omega))$, it has a component \mathcal{N}'' which is a Seifert fiber space with γ, γ' lying in different components of $\partial\mathcal{N}''$, and γ, γ' cobound an annulus in \mathcal{N}'' .

The case $\mathcal{N}'' \approx \mathbf{T}^2 \times \mathbf{I}$ is discarded since Ω has at least two components. So with Lemma II.2.8 of [JS] γ, γ' are homotopic to regular fibers. Now let $\gamma'' \in \sigma\mathcal{T}_j \cap \mathcal{T}''$ for $\mathcal{T}'' \subset \Omega$, such that γ, γ'' cobound an innermost annulus $\mathcal{B} \neq \mathcal{A}$ in $\sigma\mathcal{T}_j$. The same argument as above shows that \mathcal{N}' contains \mathcal{N}''' which is a Seifert fiber space and if $\mathcal{T} \neq \mathcal{T}''$, γ is homotopic to a regular fiber of \mathcal{N}''' . Hence $\mathcal{T}' \neq \mathcal{T} \neq \mathcal{T}''$ is impossible cause otherwise the Seifert fibrations of $\mathcal{N}'', \mathcal{N}'''$ would both extend to $\mathcal{N}'' \cup \mathcal{N}''' \cup \text{int}(V(\mathcal{T}))$ and $\Omega \setminus \mathcal{T}$ would be a smaller JSJ decomposition.

In summary when $\sigma\mathcal{T}_j \cap \Omega$ consists in simple closed curves one can find $\mathcal{T} \subset \Omega$ and two curves $\gamma, \gamma' \subset \mathcal{T}$ which cobound in $\sigma\mathcal{T}_j$ an innermost annulus \mathcal{A} . The curves γ, γ' cobound an annulus \mathcal{B} in \mathcal{T} . Modify Ω by changing \mathcal{T} into $\mathcal{T} \cup \mathcal{A} \setminus \text{int}(\mathcal{B})$ and \mathcal{T}_j into $\mathcal{T}_j \cup \sigma\mathcal{B} \setminus \text{int}(\sigma\mathcal{A})$, and perform a small isotopy so that the number of components in $\Omega \cap \sigma\Omega$ decreases (cf. figure 5). Pursue this process until $\Omega \cap \sigma\Omega$ has no more closed curve component.

Up to this point Ω is a JSJ decomposition of $\hat{\mathcal{N}}$; its components are 2-sided essential tori which fall in two parts: those with $\sigma\mathcal{T} \subset \Omega$ and those with $\sigma\mathcal{T} \cap \Omega = \emptyset$. For those \mathcal{T} such that $\sigma\mathcal{T} \cap \mathcal{T} = \emptyset$, $p(\mathcal{T})$ is a two-sided essential torus in \mathcal{M}_i . For those \mathcal{T} such that $\sigma\mathcal{T} = \mathcal{T}$ and σ reverses orientation on \mathcal{T} , $p(\mathcal{T})$ is a two-sided Klein bottle in \mathcal{M}_i . For those \mathcal{T} such that $\sigma\mathcal{T} = \mathcal{T}$ and σ preserves orientation of \mathcal{T} , $p(\mathcal{T})$ is a one-sided torus. In such case consider a regular neighborhood $V(\mathcal{T})$

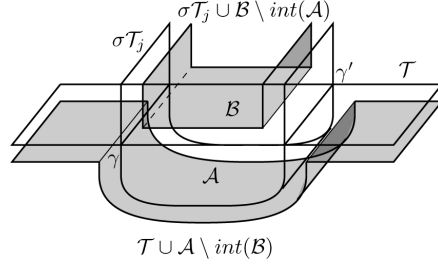


FIGURE 5. By considering small enough collar neighborhood $N(\mathcal{T})$ of \mathcal{T} and $N(\sigma\mathcal{T}_j)$ of $\sigma\mathcal{T}_j$ in a subdivision of the triangulation of \mathcal{N} , one can deform by isotopy the tori $\mathcal{T} \cup \mathcal{A} \setminus \text{int}(\mathcal{B})$ and $\sigma\mathcal{T} \cup \mathcal{B} \setminus \text{int}(\mathcal{A})$ so that their number of intersections decreases.

of \mathcal{T} in \mathcal{N} with $\sigma V(\mathcal{T}) = V(\mathcal{T})$ and change \mathcal{T} in Ω by a component of $\partial V(\mathcal{T})$.

Finally for any \mathcal{T} such that $\sigma\mathcal{T} \cap \Omega = \emptyset$: by the characteristic pair theorem (cf. [JS]) $\sigma\mathcal{T}$ is parallel to some $\mathcal{T}' \subset \Omega$. If $\mathcal{T}' = \mathcal{T}$, then \mathcal{T} and $\sigma\mathcal{T}$ cobound $\mathbf{T}^2 \times \mathbf{I}$ preserved by σ or \mathcal{N} is a torus bundle modeled on Sol (cf. Theorem 5.3, [Sc]). If $\mathcal{T}' \neq \mathcal{T}$; note that $\sigma\mathcal{T}' = \mathcal{T}$ and change in Ω the component \mathcal{T}' by $\sigma\mathcal{T}$. At the end of the process Ω and $\Xi = p(\Omega)$ are coherent JSJ decompositions of \mathcal{N} and \mathcal{M} . \square

- Apply algorithm **Top3** to find coherent JSJ decompositions Ω_i of the \mathcal{N}_i 's and Ξ_i of the \mathcal{M}_i 's.

In the following we denote $\Omega = \bigcup_i \Omega_i$ and $\Xi = \bigcup_i \Xi_i$. Note that the involution σ naturally acts on $\mathcal{N} \setminus \Sigma \sqcup \Omega \sqcup \sigma\Omega$ by permuting connected components and one defines the σ -equivariant maps $p : \mathcal{N} \setminus (\Sigma \sqcup \Omega \sqcup \sigma\Omega) \longrightarrow \mathcal{M} \setminus (\Pi \sqcup \Xi)$ and $p : \Sigma \sqcup \Omega \sqcup \sigma\Omega \longrightarrow \Pi \sqcup \Xi$ by restriction of $p : \mathcal{N} \longrightarrow \mathcal{M}$.

Lemma 2.5 (Algorithm **Top4**). *There is an algorithm which checks for each component \mathcal{Q} of $\mathcal{N} \setminus \Sigma \sqcup \Omega \sqcup \sigma\Omega$, whether $\hat{\mathcal{Q}}$ is a Seifert fiber space and if so returns a Seifert fibration of $\hat{\mathcal{Q}}$ by mean of a set of invariants.*

Proof. The algorithm is given in algorithm 8.1, [JT]; it implicitly produces a set of invariants. \square

- Apply algorithm **Top4** to decide which pieces of $\mathcal{N} \setminus \Sigma \sqcup \Omega \sqcup \sigma\Omega$ are punctured Seifert fiber spaces and for each find a Seifert fibration by mean of a set of invariants.

2.3. Step 3: Produce graph of groups splittings for $\pi_1(\mathcal{M})$ and $\pi_1(\mathcal{N})$. We now focus on how $\pi_1(\mathcal{M})$ and $\pi_1(\mathcal{N})$ can be given in a constructive way by finite sets of data. This can be achieved by constructing graphs of group related to the topological decompositions of \mathcal{M}

and \mathcal{N} where vertex and edge groups are given by finite presentations.

We first need to establish the following algorithms which will be useful in the remaining of this part. We say that a 3-manifold \mathcal{V} has *incompressible boundary* if for any $\mathcal{T} \subset \partial\mathcal{V}$, the inclusion $i : \mathcal{T} \hookrightarrow \mathcal{V}$ induces a monomorphism $i_* : \pi_1(\mathcal{T}) \longrightarrow \pi_1(\mathcal{V})$. For a 3-manifold with incompressible boundary a peripheral subgroups system is a collection of the embeddings of the π_1 of components of $\partial\mathcal{V}$ into $\pi_1(\mathcal{V})$ induced by inclusions; each embedding in a peripheral subgroups system of $\pi_1(\mathcal{V})$ is well defined only up to conjugacy in $\pi_1(\mathcal{V})$.

Lemma 2.6 (Basic algorithms in π_1 of the pieces). *Let \mathcal{V} be a 3-manifold given by a triangulation, $q : \mathcal{W} \longrightarrow \mathcal{V}$ be the orientation cover, $V = \pi_1(\mathcal{V})$ and $W = \pi_1(\mathcal{W})$ seen as a subgroup of V (when \mathcal{V} is orientable $q : \mathcal{W} \longrightarrow \mathcal{V}$ is an homeomorphism).*

- (i) (Finite presentations). *One can algorithmically produce finite presentations $\langle S | R \rangle$ of V and $\langle S' | R' \rangle$ of W with S' a set of words on $S \cup S^{-1}$.*
- (ii) (Algorithm $\mathbf{Gwp}(W, V)$). *Given a word w on $S \cup S^{-1}$ one can decide whether $w \in W$ and if so produce a word w' on $S' \cup S'^{-1}$ which represents the same element.*

In the following $\partial\mathcal{V}$ is incompressible and consist in \mathbf{S}^2 , \mathbf{P}^2 , \mathbf{T}^2 , \mathbf{K}^2 .

- (iii) (Peripheral subgroups system). *One can construct peripheral subgroups system $(V_i)_{i=1\dots p}$ of V (respectively $(W_j)_{j=1\dots q}$ of W) by canonical finite presentations with generators words on $S \cup S^{-1}$ (respectively on $S' \cup S'^{-1}$), represented by loops in $\partial\mathcal{V}$ (respectively $\partial\mathcal{W}$) and such that for $i = 1 \dots p$, $V_i \cap W = W_i$.*
- (iv) (Generalized word problem for boundary subgroups). *Here we moreover suppose that \mathcal{V} is geometrizable. Given peripheral subgroups systems $(V_i)_i$ of V and $(W_j)_j$ of W , and w a word on $S \cup S^{-1}$ (respectively on $S' \cup S'^{-1}$), one can decide whether $w \in V_i$ (respectively $w \in W_j$) and if so find a word w' on generators of V_i (respectively of W_j) which represents the same element in V (respectively in W).*

Proof. In case \mathcal{V} is orientable $q : \mathcal{W} \longrightarrow \mathcal{V}$ is an homeomorphism, and simply skip in the lines of the proof all assertions involving \mathcal{W} or W .

Proof of (i) and (ii). As in the first step of application of algorithm $\mathbf{Top1}$ in the proof of Lemma 2.2, construct from a triangulation of \mathcal{V} a pl-ball \mathcal{B} with pl-identification f of triangles of $\partial\mathcal{B}$ with quotient manifold homeomorphic to \mathcal{V} . Let D_f , the *domain* of f , be the union of all triangles in $\partial\mathcal{B}$ identified by f . Choose a point $*$ in \mathcal{B} ; for any pl-triangle δ in D_f choose a loop δ_1 in \mathcal{B} from $*$ to the center of gravity δ_* of δ made pl by subdividing the triangulation, and similarly a pl-loop δ_2 in \mathcal{B} from $f(\delta_*)$ to $*$ and consider the pl-loop based in $*$, $l(\delta) = \delta_1\delta_2$ (cf. figure 6); let $\lambda S = \{l(\delta), \delta \in D_f\}$ be the finite set of all pl-loops based in $*$ obtained in this way. Given any pl-loop l in \mathcal{V} based in $*$

there is an algorithm which homotopically changes, with $*$ fixed, the $*$ -loop l into a product of elements of λS : simply deform slightly l so that it becomes transverse with $\partial\mathcal{B}$ then look at the triangles in D_f in which l passes successively through $\partial\mathcal{B}$ to write it down as a product of elements of λS . In particular λS is a set of representatives of a generating set S of $V \simeq \pi_1(\mathcal{V}, *)$.

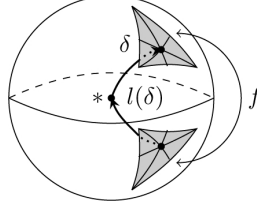


FIGURE 6. The loop $l(\delta)$ based in $*$ in \mathcal{V} defined by a triangle δ with identification in $\partial\mathbf{B}^3$; it yields a generator of $\pi_1(\mathcal{V}, *)$.

Use the algorithm of [RS] to compute a finite presentation $\langle T|U \rangle$ of $\pi_1(\mathcal{V}, *)$ from the triangulation of \mathcal{V} . It considers the 1-skeleton K of the triangulation and constructs a spanning tree T of K ; for any edge e in $K \setminus T$ let l_1 be the simple pl-loop in T from $*$ to the origin of e and l_2 the simple pl-loop in T from the extremity of e to $*$ and let $l_e = l_1 e l_2$ a $*$ -loop which passes through e . The set T of generators is represented by the set of all $*$ -loops l_e constructed in this way. For any pl-loop l based in $*$ one algorithmically constructs a product of the l_e , $e \in K \setminus T$, homotopic to l with $*$ fixed, by reading the successive edges of $K \setminus T$ which appears in l .

Use the two process described above to write down elements $s \in S$ as words $T(s)$ on $T \cup T^{-1}$, and elements $t \in T$ as words $S(t)$ on $S \cup S^{-1}$ and apply the following sequence of Tietze transformations (cf. [MKS]) to the presentation $\langle T|U \rangle$:

- add generator s and relation $s = T(s)$ for all $s \in S$, to obtain $\langle S \cup T|U \cup U_1 \rangle$
- add relations $t = S(t)$ for all $t \in T$, to obtain $\langle S \cup T|U \cup U_1 \cup U_2 \rangle$,
- use relations in U_2 to change each relation in $U \cup U_1$ and express it on the alphabet S , to obtain $\langle S \cup T|U' \cup U'_1 \cup U_2 \rangle$,
- delete generators in T and relations in U_2 , to obtain $\langle S|U' \cup U'_1 \rangle$,

which finally yields a finite presentation of V onto generating set S and proves the first assumption in (i). This presentation has a large number of generators and can be easily improved by identifying generators using a splitting of D_f into connected surfaces on which f provides homeomorphisms.

Among the set S of generators one can decide which reverses orientation and which doesn't: for the loop $l(\delta)$ reverses orientation if and

only if $f|_\delta : \delta \longrightarrow f(\delta)$ reverses the orientation induced on $\delta, f(\delta)$ by the orientation of \mathcal{B} . Hence given a word on $S \cup S^{-1}$ one can decide if it represents an element in W simply by counting whether it has an even occurrence of orientation reversing generators or not. Consider the set S' of words of one of the form: s , or $s's''$ or $s'ss'^{-1}$ for any s orientation preserving, any s', s'' orientation reversing, elements of $S \cup S^{-1}$. Each word on $S \cup S^{-1}$ with an even number of occurrence of orientation reversing element can be easily written (in linear time) as a word on $S' \cup S'^{-1}$ for example by the deterministic pushdown automata in figure 7. This shows that S' generates W and proves (ii).

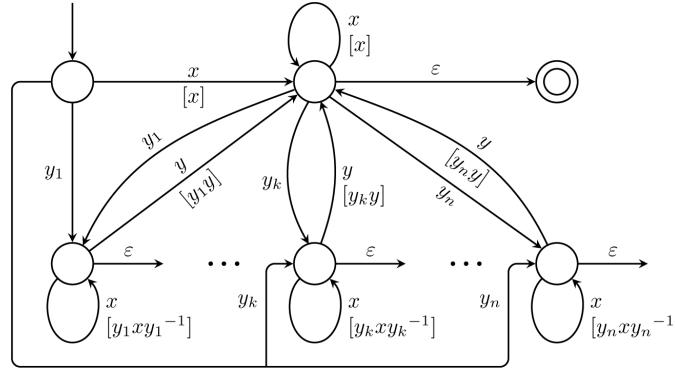


FIGURE 7. A pushdown automata which given a word on generators $S \cup S^{-1}$ of V decides whether it represents an element in the index 2 subgroup W and if so returns in the stack a representative as a word on the generators $S' \cup S'^{-1}$ of W described above. Elements of $S \cup S^{-1}$ are denoted x if they lie in W and y, y_1, \dots, y_n otherwise; ε is the empty string; in bracket the element of $S' \cup S'^{-1}$ pushed on the stack.

Finally apply a process such as Reidemeister-Schreier (cf. [MKS, Jo]) to build a finite presentation $\langle T' | U' \rangle$ of W , then express each generator in S' onto a word on $T' \cup T'^{-1}$ and each generator in T' as a word on $S' \cup S'^{-1}$ and apply Tietze transformations as above to obtain a finite presentation $\langle S' | R' \rangle$ with generators S' for W . This achieves the proof of (i).

Proof of (iii). First note that since $q : \mathcal{W} \longrightarrow \mathcal{V}$ induces a monomorphism on fundamental groups, whenever \mathcal{V} has boundary incompressible so has \mathcal{W} . The set of pl-triangles in $\partial\mathcal{B} \setminus D_f$ together with f provides a triangulation of $\partial\mathcal{M}$. Use it to compute the Euler characteristic χ and check orientability for each component of $\partial\mathcal{M}$; it determines their homeomorphism classes \mathbf{S}^2 , \mathbf{P}^2 , \mathbf{T}^2 or \mathbf{K}^2 depending on whether $\chi = 2$, $\chi = 1$, or $\chi = 0$ and orientability, or $\chi = 0$ and non-orientability. Then by a favorite trick (such as representing each surface in $\partial\mathcal{M}$ by a pl-disk with identification on its boundary edges and deform to get one of 4 standard models, cf. [ST]) find for

each surface $\not\approx \mathbf{S}^2$ in $\partial\mathcal{M}$ a family of $2 - \chi$ pl -curves which represent generators of one of the canonical presentations $\langle a | a^2 = 1 \rangle$, $\langle a, b | [a, b] = 1 \rangle$, or $\langle a, b | aba^{-1} = b^{-1} \rangle$ of \mathbb{Z}_2 , $\mathbb{Z} \oplus \mathbb{Z}$, and $\mathbb{Z} \rtimes \mathbb{Z}$. Finally use the algorithm described in the proof of (i) above to write down the generators on the alphabet $S \cup S^{-1}$. It defines a peripheral subgroups system of V .

An element in a peripheral subgroup of V can be orientation preserving/reversing as an element of V , but also as an element of the surface group; note that since surfaces arise from the boundary those two notions coincide. The boundary of \mathcal{W} is $q^{-1}(\partial\mathcal{V})$ and its components are related to those of $\partial\mathcal{V}$:

- any $\mathbf{S}^2, \mathbf{P}^2 \hookrightarrow \partial\mathcal{V}$ lifts to $\mathbf{S}^2 \hookrightarrow \partial\mathcal{W}$. Since $\pi_1(\mathbf{S}^2) = \{1\}$, the embedding $\pi_1(\mathbf{S}^2) \rightarrow \pi_1(\mathcal{W})$ is well defined.
- any $\mathbf{T}^2 \hookrightarrow \partial\mathcal{V}$ lifts to two components $\mathbf{T}^2 \hookrightarrow \partial\mathcal{W}$. The embedding $\phi : \pi_1(\mathbf{T}^2) \rightarrow V$ has image in W ; let α, β be two based loops in \mathbf{T}^2 which represent generators $[\alpha], [\beta]$ of $\pi_1(\mathbf{T}^2)$, $i : \mathbf{T}^2 \hookrightarrow \mathcal{V}$, and $a = \phi([\alpha])$, $b = \phi([\beta])$; the loops $i(\alpha), i(\beta)$ are orientation preserving in \mathcal{V} and lift to loops α^+, β^+ and α^-, β^- lying in the two $\mathbf{T}^2 \hookrightarrow \partial\mathcal{W}$ where they both represent a basis of $\pi_1(\mathbf{T}^2)$. One defines the respective embeddings $\phi_+ : \pi_1(\mathbf{T}^2) \rightarrow W$ by $\phi_+([\alpha^+]) = a$, $\phi_+([\beta^+]) = b$ and $\phi_- : \pi_1(\mathbf{T}^2) \rightarrow W$, by $\phi_-([\alpha^-]) = vav^{-1}$, $\phi_-([\beta^-]) = vbv^{-1}$ for some arbitrary element v of $V \setminus W$.
- any $\mathbf{K}^2 \hookrightarrow \partial\mathcal{N}$ lifts to a $\mathbf{T}^2 \hookrightarrow \partial\mathcal{W}$. Let $\pi_1(\mathbf{K}^2) \rightarrow V$ be the embedding found above, and let α, β be based loops in \mathbf{K}^2 such that $\phi([\alpha]) = a$ and $\phi([\beta]) = b$ for canonical generators a, b of $\mathbb{Z} \rtimes \mathbb{Z}$ as in the presentation above. Let $i \hookrightarrow \mathcal{V}$; $i(\alpha), i(\beta)$ are respectively orientation reversing and orientation preserving loops in \mathcal{N} , then consider the two loops α_2 and β_1 , respective lifts of $i(\alpha)^2$ and $i(\beta)$ in $\partial\mathcal{W}$; they represent generators of $\pi_1(\mathbf{T}^2)$. Let the embedding $\phi' : \pi_1(\mathbf{T}^2) \rightarrow W$ be defined by $\phi'([\alpha_2]) = a^2$ and $\phi'([\beta_1]) = b$.

We finally construct a peripheral subgroup system in W , which proves (iii). By construction whenever V_i is a peripheral subgroup of V , $V_i \cap W$ is a peripheral subgroup W_i of W .

Proof of (iv). Since \mathcal{V} is geometrizable one has a solution to the word problem in V (cf. [CEHLPT]). In particular in case of the peripheral subgroup $\{1\}$ coming from a \mathbf{S}^2 component one can solve the generalized word problem.

Suppose first that \mathcal{V} is orientable. Then any peripheral subgroup $V_1 \neq \{1\}$ comes from a torus $\mathbf{T}^2 \hookrightarrow \partial\mathcal{V}$. Consider an homeomorphic copy \mathcal{V}' of \mathcal{V} , and the double $2\mathcal{V} = \mathcal{V} \sqcup_{\mathbf{T}^2} \mathcal{V}'$ of \mathcal{V} along the boundary component $\mathbf{T}^2 \hookrightarrow \partial\mathcal{V}$. The same argument as in Lemma 1.2 of [Pr] shows that $2\mathcal{V}$ is geometrizable, hence $\pi_1(2\mathcal{V})$ has solvable word problem. The group $\pi_1(2\mathcal{V})$ splits into the amalgam $V *_{V_1} V'$ equipped with the isomorphism $v \in V \mapsto v' \in V'$ which restricts to identity on the subgroup $V_1 \simeq \mathbb{Z} \oplus \mathbb{Z}$. Let v be an element of V given by a word on

$S \cup S^{-1}$, then by the normal form theorem for amalgams (cf. [MKS]), $v \in V_1$ if and only if $v^{-1}v' = 1$ in $\pi_1(2\mathcal{V})$, which can be checked using the solution to word problem in $\pi_1(2\mathcal{V})$. If yes enumerate elements in V_1 as words on the generators found in (iii) and for each use a solution to the word problem in V to decide whether it equals v , to finally write down v as a word on generators of V_1 ; this naive process can be improved using the quasi-convexity of V_1 in V (cf. [CEHLPT]). (Note that in case \mathcal{V} is an orientable piece coming from a JSJ decomposition of a closed irreducible 3-manifold –as occurs in our context– a far more efficient solution is given by proposition 4.2 of [Pr]). In particular this solves the generalized word problem for peripheral subgroups in W .

Suppose now that \mathcal{V} is non-orientable. Let $V_1 \neq \{1\}$ be a peripheral subgroup of V ; V_1 is isomorphic either to \mathbb{Z}_2 , $\mathbb{Z} \oplus \mathbb{Z}$ or $\mathbb{Z} \rtimes \mathbb{Z}$. If $V_1 \simeq \mathbb{Z}_2$ is generated by a then v lies in V_1 if and only if v commutes with a (cf. [Sw]) that one decides using a solution to the word problem, and if so write v as a word -1 or a on generator of V_1 . If $V_1 \simeq \mathbb{Z} \oplus \mathbb{Z}$, then $V_1 \subset W$ is also, by (iii), a peripheral subgroup of W . One decides first using (ii) whether $v \in W$, and if so solves the problem reduced in W by the above solution in W . If $V_1 \simeq \mathbb{Z} \rtimes \mathbb{Z}$; let $t \in V_1 \setminus (V_1 \cap W)$ be an o.r. element in V_1 provided by (iii). One decides whether $v \in W$ other not; then using the above solution in W , one decides whether, v in the former case, and vt in the latter case, lies in the peripheric subgroups $V_1 \cap W$ of W . If yes it provides a word on generators of V_1 equal to v . In any case this solves the generalized word problem in a peripheral subgroup of V and proves (iv). \square

We now turn to the description of $\pi_1(\mathcal{M})$ and $\pi_1(\mathcal{N})$ as graphs of groups associated to their topological decompositions obtained in step 2.

Recall that a *graph of group* \mathbf{X} consists in (cf. [Se]):

- a non-empty connected finite oriented graph \mathbf{X} ; let $V\mathbf{X}$, $E\mathbf{X}$ denote respectively the vertex and edge sets of \mathbf{X} , for all $e \in E\mathbf{X}$, \bar{e} denotes the opposite edge of e , and $t(e) \in V\mathbf{X}$ denotes the extremity of e ; the edge e has origin $t(\bar{e})$ and extremity $t(e)$,
- two families of *vertex groups* $G(v)$ for any $v \in V\mathbf{X}$ and *edge groups* $G(e)$ for any $e \in E\mathbf{X}$, with $G(\bar{e}) = G(e)$ for all $e \in E\mathbf{X}$,
- a family of monomorphisms $\phi_e : G(e) \longrightarrow G(t(e))$ for any $e \in E\mathbf{X}$.

Graphs of groups come equipped with the notion of *fundamental group* of graph of group \mathbf{X} (cf. [Se, Ba]), that we now introduce. An \mathbf{X} -path of length $n \in \mathbb{N}$ is a finite sequence $(g_0, e_1, g_1, \dots, e_n, g_n)$ such that $\forall i = 1 \dots n - 1$, one has $t(e_i) = t(\bar{e}_{i+1})$, $g_i \in G(t(e_i))$, $g_0 \in G(t(\bar{e}_1))$ and $g_n \in G(t(e_n))$. We denote by $\pi(\mathbf{X})$ the set of \mathbf{X} -paths. An \mathbf{X} -path is *reduced* if it does not contain a subsequence $(\dots, e, \phi_e(g), \bar{e}, \dots)$ for some $e \in E\mathbf{X}$, $g \in G(e)$; any \mathbf{X} -path can be

transformed into a reduced \mathbf{X} -path by changing each subsequence of the above form using relations:

$$(\dots, g', e, \phi_e(g), \bar{e}, g'', \dots) \equiv (\dots, g' \phi_{\bar{e}}(g) g'', \dots) \quad (*)$$

for any $e \in EX$ and $g \in G(e)$; moreover any two reductions of an \mathbf{X} -path must have the same length. The set $\pi(\mathbf{X})$ comes equipped with a partially defined *concatenation product*: $(\dots, e_1, g_1)(g_2, e_2, \dots) = (\dots, e_1, g_1 g_2, e_2, \dots)$ anytime $t(e_1) = t(\bar{e}_2)$. Let $x \in VX$; an (\mathbf{X}, x) -loop is an \mathbf{X} -path as above such that $t(\bar{e}_1) = t(e_n) = x$. The concatenation product is well defined on (\mathbf{X}, x) -loops and compatible with relations $(*)$; the equivalence classes of (\mathbf{X}, x) -loops with respect to relation $(*)$ inherits a group structure, and we denote this group by $\pi_1(\mathbf{X}, x)$, the *fundamental group* of \mathbf{X} based in x . Each element of $\pi_1(\mathbf{X}, x)$ can be represented by a reduced (\mathbf{X}, x) -loop which allows to define its length. The isomorphism class of $\pi_1(\mathbf{X}, x)$ does not depend on $x \in VX$ and will be denoted by $\pi_1(\mathbf{X})$.

Given a 3-manifold \mathcal{V} and a two-sided compact incompressible surface Φ in \mathcal{V} there is a usual way to define a *graph of group \mathbf{V} related to (\mathcal{V}, Φ)* with $\pi_1(\mathbf{V}) \simeq \pi_1(\mathcal{V})$. Consider the interior $N(\Phi)$ of a regular neighborhood of Φ in \mathcal{V} . Vertices v_i (respectively edges e_j) of \mathbf{V} are in 1-1 correspondence with components \mathcal{V}_i of $\mathcal{V} \setminus N(\Phi)$ (respectively with components \mathcal{T}_j of Φ), vertex groups (respectively edge groups) are $G(v_i) = \pi_1(\mathcal{V}_i)$ (respectively $G(e_j) = \pi_1(\mathcal{T}_j)$). The embedding of Φ in \mathcal{V} defines for each $\mathcal{T}_j \in \Phi$ two embeddings f_j^+, f_j^- of \mathcal{T}_j into the boundary of components $\mathcal{V}_i, \mathcal{V}_k$ of $\mathcal{V} \setminus N(\Phi)$ (with eventually $i = k$) inducing monomorphisms g_j^+, g_j^- on their π_1 . With the identifications above one defines $\phi_{e_j} = g_j^+ : G(e_j) \rightarrow G(t(e_j))$ and $\phi_{\bar{e}_j} = g_j^- : G(\bar{e}_j) \rightarrow G(t(\bar{e}_j))$. This defines a graph of group which depends on all monomorphisms g_j^+, g_j^- despite its isomorphism class of fundamental group $\pi_1(\mathbf{V})$ doesn't. One proves applying the Seifert-Van Kampen theorem that indeed $\pi_1(\mathbf{V}) \simeq \pi_1(\mathcal{V})$.

Given topological decompositions $\Sigma \sqcup \Omega$ of \mathcal{N} and $\Pi \sqcup \Xi$ of \mathcal{M} we consider a graph of group \mathbf{M} related to $(\mathcal{M}, \Pi \sqcup \Xi)$ and a graph of group related to $(\mathcal{N}, \Sigma \sqcup \Omega \sqcup \sigma\Omega)$ (rather than on $(\mathcal{N}, \Sigma \sqcup \Omega)$). This last graph of group slightly differs from it related to $(\mathcal{N}, \Sigma \sqcup \Omega)$ in that it is possibly non-minimal (*i.e.* it may contain an edge e with $t(e) \neq t(\bar{e})$ and φ_e is onto); this prizes for the gain of a *covering* of graphs of groups. More precisely, by *coherent graph of group decompositions* for $\pi_1(\mathcal{M})$ and $\pi_1(\mathcal{N})$ we mean:

- a graph of groups \mathbf{M} related to $(\mathcal{M}, \Pi \sqcup \Xi)$, with $\pi_1(\mathbf{M}) \simeq \pi_1(\mathcal{M})$,
- a graph of group \mathbf{N} related to $(\mathcal{N}, \Sigma \sqcup \Omega \sqcup \sigma\Omega)$, with $\pi_1(\mathbf{N}) \simeq \pi_1(\mathcal{N})$,
- a *covering* $p : \mathbf{N} \rightarrow \mathbf{M}$, that is a collection of:
 - a map of graphs $p : \mathbf{N} \rightarrow \mathbf{M}$ from the underlying graph \mathbf{N} of \mathbf{N} to the underlying graph \mathbf{M} of \mathbf{M} , induced by $p : \mathcal{N} \rightarrow \mathcal{M}$,

- two families of monomorphisms $p_v : G(v) \longrightarrow G(p(v))$, $v \in VN$ and $p_e : G(e) \longrightarrow G(p(e))$, $e \in EN$,
- a collection of elements $\mu(e)$, $e \in EN$, with $\mu(e) \in G(t(p(e)))$ such that if ad_e is the automorphism of $G(t(p(e)))$ defined by $\forall g \in G(t(p(e))), ad_e(g) = \mu(e) g \mu(e)^{-1}$, the following diagram commutes:

$$\begin{array}{ccc} G(e) & \xrightarrow{\phi_e} & G(t(e)) \\ p_e \downarrow & & \downarrow p_{t(e)} \\ G(p(e)) & \xrightarrow{ad_e \circ \phi_{p(e)}} & G(t(p(e))) \end{array}$$

- a map $p_{\#} : \pi(\mathbf{N}) \longrightarrow \pi(\mathbf{M})$ defined by:

$$p_{\#}(g_0, e_1, g_1, e_2, \dots, e_n, g_n) = (g'_0, p(e_1), g'_1, p(e_2), \dots, p(e_n), g'_n)$$

where:

$$\forall i = 0, \dots, n, g'_i = \begin{cases} p_{t(\bar{e}_1)}(g_0) \mu(\bar{e}_1)^{-1} & \text{for } i = 0 \\ \mu(e_i) p_{t(e_i)}(g_i) \mu(\bar{e}_{i+1})^{-1} & \text{for } i \neq 0, n \\ \mu(e_n) p_{t(e_n)}(g_n) & \text{for } i = n \end{cases}$$

and for any $x \in VM$, $\tilde{x} \in p^{-1}(x)$, $p_{\#}$ induces a monomorphism:

$$p_* : \pi_1(\mathbf{N}, \tilde{x}) \longrightarrow \pi_1(\mathbf{M}, x) .$$

One may refer to [Ba] for a general definition of covering of graphs of groups; its formalism slightly differs from our which turns to be more practical in the present context though less general; we won't need to relate to the definition of [Ba] in our purpose.

Lemma 2.7 (Algorithm for graphs of groups). *One can algorithmically produce coherent graphs of groups decomposition \mathbf{N} and \mathbf{M} for $\pi_1(\mathcal{N})$ and $\pi_1(\mathcal{M})$ related to the topological decompositions $(\mathcal{N}, \Sigma \sqcup \Omega \sqcup \sigma\Omega)$ and $(\mathcal{M}, \Pi \sqcup \Xi)$ and computable covering of graphs of groups $p : \mathbf{N} \longrightarrow \mathbf{M}$ and induced monomorphism $p_* : \pi_1(\mathbf{N}, \tilde{x}) \longrightarrow \pi_1(\mathbf{M}, x)$ for any vertex x of \mathbf{M} , and $\tilde{x} \in p^{-1}(x)$.*

Proof. The graph of group \mathbf{M} is obtained from the topological decomposition of \mathcal{M} along $\Pi \sqcup \Xi$ performed by algorithms **Top2**, **Top3** and from finite presentations of the fundamental groups of the pieces obtained in Lemma 2.6.(i) together with their peripheral subgroups systems given by algorithm of Lemma 2.6.(iii).

If M denotes the underlying graph of \mathbf{M} , VM is in 1-1 correspondence with connected components of $\mathcal{M} \setminus (\Pi \sqcup \Xi)$ and EM is in 1-1 correspondence with components of $\Pi \cup \Xi$. For each $v \in VM$, $G(v)$ is the fundamental group of the corresponding component of $\mathcal{M} \setminus (\Pi \sqcup \Xi)$, and for each $e \in EM$, $G(e) = \mathbb{Z}_2$, $\mathbb{Z} \oplus \mathbb{Z}$, or $\mathbb{Z} \rtimes \mathbb{Z}$ according to the associated component is homeomorphic to \mathbf{P}^2 , \mathbf{T}^2 or \mathbf{K}^2 . The monomorphisms $\phi_e : G(e) \longrightarrow G(t(e))$ are induced by the sewing maps together with peripheral subgroups systems in the vertex groups.

Now that a graph of group \mathbf{M} with $\pi_1(\mathbf{M}) \simeq \pi_1(\mathcal{M})$ associated to the splitting of \mathcal{M} along $\Pi \sqcup \Xi$ is given we construct from \mathbf{M} a related graph of group splitting \mathbf{N} of $\pi_1(\mathcal{N})$. This graph of group \mathbf{N} is related to the topological decomposition of \mathcal{N} along $\Sigma \sqcup \Omega \sqcup \sigma\Omega$: despite we only focus on the graph of group, keep in mind in the line of the proof that the construction of \mathbf{N} encodes how \mathcal{N} and $\Sigma \sqcup \Omega \sqcup \sigma\Omega$ are constructed by gluing the orientation covers of components of $\mathcal{M} \setminus (\Pi \sqcup \Xi)$ and yields a related graph of group.

Partition \mathbf{VM} into

$$\mathbf{VM} = \mathbf{VM}^+ \sqcup \mathbf{VM}^-, \quad \mathbf{VM}^+ = \{v_1, \dots, v_q\}, \quad \mathbf{VM}^- = \{v_{q+1}, \dots, v_r\}$$

where \mathbf{VM}^+ are those vertices coming from oriented components and \mathbf{VM}^- those coming from non-orientable components of $\mathcal{M} \setminus (\Pi \sqcup \Xi)$ (orientability of the pieces in $\mathcal{M} \setminus \Pi \sqcup \Xi$ can be algorithmically checked from their triangulations). For any $v \in \mathbf{VM}^-$, $G(v) \simeq \pi_1(\mathcal{M}_i)$ naturally comes with an index two subgroup described by Lemma 2.6.(i), that we denote by $H(v^+)$, of orientation preserving elements; choose for any $v \in \mathbf{VM}^-$ an arbitrary element $\mu(v)$ in $G(v) \setminus H(v^+)$, it defines an automorphism ad_v of $H(v^+)$ by $\forall h \in H(v^+)$, $ad_v(h) = \mu(v) h \mu(v)^{-1}$. Moreover the choice of $\mu(v)$ defines a peripheral subgroups system of $H(v^+)$ as in Lemma 2.6.(iii) (cf. proof of 2.6.(iii)).

Similarly partition \mathbf{EM} into

$$\mathbf{EM} = \mathbf{EM}^+ \sqcup \mathbf{EM}^-, \quad \mathbf{EM}^+ = \{e_1, \dots, e_s\}, \quad \mathbf{EM}^- = \{e_{s+1}, \dots, e_t\}$$

where \mathbf{EM}^+ are edges associated to \mathbf{T}^2 and \mathbf{EM}^- edges associated to \mathbf{P}^2 or \mathbf{K}^2 ; note that $t(\mathbf{EM}^-) \subset \mathbf{VM}^-$ and $e \in \mathbf{EM}^-$ if and only if $\bar{e} \in \mathbf{EM}^-$.

One constructs the graph \mathbf{N} by picking $2r - q$ vertices and $2t - s$ edges:

$$\begin{aligned} \mathbf{VN} &= \{v_1^+, \dots, v_q^+, v_1^-, \dots, v_q^-, v_{q+1}^+, \dots, v_r^+\}, \\ \mathbf{EN} &= \{e_1^+, \dots, e_s^+, e_1^-, \dots, e_s^-, e_{s+1}^+, \dots, e_t^+\} \end{aligned}$$

and setting:

$$\begin{aligned} \forall i = 1 \dots t, \quad t(e_i^+) &= t(e_i)^+ \\ \forall i = 1 \dots s, \quad t(e_i^-) &= \begin{cases} t(e_i)^+ & \text{whenever } t(e_i) \in \mathbf{VM}^- \\ t(e_i)^- & \text{whenever } t(e_i) \in \mathbf{VM}^+. \end{cases} \end{aligned}$$

Define the map of graphs $p : \mathbf{N} \longrightarrow \mathbf{M}$ by $p(v^\pm) = v$ and $p(e^\pm) = e$.

The vertex groups $(H(v))_{v \in \mathbf{VN}}$ of \mathbf{N} are defined by:

$$\begin{aligned} \forall v \in \mathbf{VM}^+, \quad H(v^+) &= H(v^-) = G(v) \\ \forall v \in \mathbf{VM}^-, \quad H(v^+) &\triangleleft_2 G(v) \end{aligned}$$

where $H(v^+) \triangleleft_2 G(v)$ is the subgroup of orientation preserving elements discussed above. The edge subgroups $(H(e))_{e \in \mathbf{EN}}$ of \mathbf{N} are

defined by:

$$\begin{aligned} \forall e \in EM^+, \quad H(e^+) = H(e^-) = G(e) = \mathbb{Z} \oplus \mathbb{Z} \\ \forall e \in EM^-, \quad H(e^+) \triangleleft_2 G(e) \text{ and } H(e^+) \simeq \begin{cases} \{1\} & \text{if } G(e) = \mathbb{Z}_2 \\ \mathbb{Z} \oplus \mathbb{Z} & \text{if } G(e) = \mathbb{Z} \rtimes \mathbb{Z} \end{cases} \end{aligned}$$

The family of monomorphisms $\phi_e : H(e) \longrightarrow H(t(e))$ for $e \in EN$ of \mathbf{N} is defined by:

- (i) Let $e \in EM^-$; necessarily $t(e) \in VM^-$. The monomorphism $\phi_e : G(e) \longrightarrow G(t(e))$ sends the index 2 subgroup $H(e^+)$ of $G(e)$ into the index 2 subgroup $H(t(e)^+)$ of $G(t(e))$. Define ϕ_{e^+} by the commutative diagram:

$$\begin{array}{ccc} H(e^+) & \xrightarrow{\phi_{e^+}} & H(t(e)^+) \\ \downarrow 2 & & \downarrow 2 \\ G(e) & \xrightarrow{\phi_e} & G(t(e)) \end{array}$$

- (ii) Let $e \in EM^+$; here $H(e^+) = H(e^-) = G(e) = \mathbb{Z} \oplus \mathbb{Z}$. There are two cases:

- (ii.a) If $t(e) \in VM^+$; Define ϕ_{e^+}, ϕ_{e^-} by the commutative diagrams:

$$\begin{array}{ccc} H(e^+) & \xrightarrow{\phi_{e^+}} & H(t(e)^+) \\ \parallel & & \parallel \\ G(e) & \xrightarrow{\phi_e} & G(t(e)) \end{array} \quad \begin{array}{ccc} H(e^-) & \xrightarrow{\phi_{e^-}} & H(t(e)^-) \\ \parallel & & \parallel \\ G(e) & \xrightarrow{\phi_e} & G(t(e)) \end{array}$$

- (ii.b) If $t(e) \in VM^-$; in that case $H(t(e)^+) \triangleleft_2 G(t(e))$ and one has the automorphism $ad_{t(e)}$ of $H(t(e)^+)$ defined above. Since $e \in EM^+$ and $t(e) \in VM^-$, one has $\phi_e(G(e)) \subset H(t(e)^+)$. Define ϕ_{e^+}, ϕ_{e^-} by the commutative diagrams:

$$\begin{array}{ccc} H(e^+) & \xrightarrow{\phi_{e^+}} & H(t(e)^+) \\ \parallel & & \downarrow 2 \\ G(e) & \xrightarrow{\phi_e} & G(t(e)) \end{array} \quad \begin{array}{ccc} H(e^-) & \xrightarrow{\phi_{e^-}} & H(t(e)^-) \\ \parallel & & \downarrow 2 \\ G(e) & \xrightarrow{ad_{t(e)} \circ \phi_e} & G(t(e)) \end{array}$$

The graph \mathbf{N} together with families of vertex groups $H(v)$, $v \in VN$, edge groups $H(e)$, $e \in EN$ and monomorphisms $\phi_e : H(e) \longrightarrow H(t(e))$, $e \in EN$, defines the graph of group \mathbf{N} . The vertex and edge groups of \mathbf{N} are subgroups of vertex and edge subgroups of \mathbf{M} , which defines the two families of monomorphisms

$$\begin{aligned} p_v : H(v) &\longrightarrow G(p(v)), v \in VN \\ p_e : H(e) &\longrightarrow G(p(e)), e \in EN. \end{aligned}$$

For each $e \in \text{EN}$ define $\mu(e) \in G(t(p(e)))$ by:

$$\text{If } e \in \text{EM}^- \quad \mu(e^+) = 1$$

$$\text{If } e \in \text{EM}^+ \quad \mu(e^+) = 1, \quad \mu(e^-) = \begin{cases} 1 & \text{if } t(e) \in \text{VM}^+ \\ \mu(t(e)) & \text{if } t(e) \in \text{VM}^- \end{cases}$$

Let ad_e be the automorphism of $G(t(p(e)))$: $ad_e(h) = \mu(e) h \mu(e)^{-1}$. By construction the following diagram commutes for all $e \in \text{EN}$:

$$\begin{array}{ccc} H(e) & \xrightarrow{\phi_e} & H(t(e)) \\ p_e \downarrow & & \downarrow p_{t(e)} \\ G(p(e)) & \xrightarrow{ad_e \circ \phi_{p(e)}} & G(t(p(e))) \end{array}$$

Consider $p_{\#} : \mathbf{N} \rightarrow \mathbf{M}$ as in the definition of covers of graphs of groups (cf p.18). Let $x \in \text{VM}$ and $\tilde{x} = x^+ \in p^{-1}(x)$; it remains to prove that $p_{\#}$ induces a monomorphism $p_* : \pi_1(\mathbf{N}, \tilde{x}) \rightarrow \pi_1(\mathbf{M}, x)$. First $p_{\#}$ induces an homomorphism $p_* : \pi_1(\mathbf{N}, \tilde{x}) \rightarrow \pi_1(\mathbf{M}, x)$ since, whenever $t(e_n) = t(\bar{e}_{n+1})$:

$$\begin{aligned} & p_{\#}(g_0, e_1, \dots, e_n, g_n) p_{\#}(h_n, e_{n+1}, \dots, e_m, g_m) \\ &= (g'_0, p(e_1), \dots, p(e_n), \mu(e_n) p_{t(e_n)}(g_n) p_{t(\bar{e}_{n+1})}(h_n) \mu(\bar{e}_{n+1})^{-1}, p(e_{n+1}), \\ & \quad \dots, p(e_m), g'_m) \\ &= (g'_0, p(e_1), \dots, p(e_n), \mu(e_n) p_{t(e_n)}(g_n h_n) \mu(\bar{e}_{n+1})^{-1}, p(e_{n+1}), \\ & \quad \dots, p(e_m), g'_m) \\ &= p_{\#}(g_0, e_1, \dots, e_n, g_n h_n, e_{n+1}, \dots, e_m, g_m) \end{aligned}$$

Secondly p_* is injective: we prove that the image of a reduced (\mathbf{N}, \tilde{x}) -loop is a reduced (\mathbf{M}, x) -loop. Clearly the image of a (\mathbf{N}, \tilde{x}) -loop is a (\mathbf{M}, x) -loop. Let $\gamma = (g_0, e_1, \dots, e_n, g_n, e_{n+1}, \dots, e_m, g_m)$ be a reduced (\mathbf{N}, \tilde{x}) -loop. Suppose that $p_{\#}(\gamma)$ is not reduced, more precisely that $p(e_n) = p(\bar{e}_{n+1})$ and $\mu(e_n) p_{t(e_n)}(g_n) \mu(\bar{e}_{n+1})^{-1}$ lies in $\phi_{p(e_n)}(G(p(e_n)))$. There are several cases to consider:

- (i) If $p(e_n) \in \text{EM}^-$; here $e_n = \bar{e}_{n+1}$, $\mu(e_n) = \mu(\bar{e}_{n+1})$ and $p_{t(e_n)}(g_n)$ lies in $\phi_{p(e_n)}(G(p(e_n)))$ if and only if $g_n \in \phi_{e_n}(H(e_n))$ (since $\phi_{e_n}(H(e_n)) = \phi_{p(e_n)}(G(p(e_n))) \cap p_{t(e_n)}(H(t(e_n)))$). In that case γ is non-reduced.
- (ii) If $p(e_n) \in \text{EM}^+$; there are two cases to consider:
 - (ii.a) if $p(t(e_n)) \in \text{VM}^+$; here $e_n \neq \bar{e}_{n+1}$ implies $t(e_n) \neq t(\bar{e}_{n+1})$ hence $e_n = \bar{e}_{n+1}$. Moreover $\mu(e_n) = \mu(\bar{e}_{n+1}) = 1$. As above γ is non-reduced.
 - (ii.b) If $p(t(e_n)) \in \text{VM}^-$; there are four cases to consider:
 - (ii.b.1) if $e_n = \bar{e}_{n+1} = p(e_n)^+$; then $\mu(e_n) = \mu(\bar{e}_{n+1}) = 1$ and as above γ is non-reduced.
 - (ii.b.2) If $e_n = \bar{e}_{n+1} = p(e_n)^-$; then $\mu(e_n) = \mu(\bar{e}_{n+1}) = \mu(t(e_n))$. Here $\mu(t(e_n)) p_{t(e_n)}(g_n) \mu(t(e_n))^{-1}$ lies in $\phi_{p(e_n)}(G(p(e_n)))$ if and only if $g_n \in \phi_{e_n}(H(e_n))$. So γ is non-reduced.

- (ii.b.3) If $e_n = p(e_n)^-$ and $\bar{e}_{n+1} = p(e_n)^+$; then $\mu(e_n) = \mu(t(e_n))$ and $\mu(\bar{e}_{n+1}) = 1$. This leads to a contradiction since $\mu(t(e_n)) p_{t(e_n)}(g_n) \notin p_{t(e_n)}(H(t(e_n)))$ while $\phi_{p(e_n)}(G(p(e_n))) \subset p_{t(e_n)}(H(t(e_n)))$.
- (ii.b.4) If $e_n = p(e_n)^+$ and $\bar{e}_{n+1} = p(e_n)^-$; one obtains a contradiction as in the latter case.

This concludes the proof. \square

- *Construct graphs of groups decomposition \mathbf{M} of $\pi_1(\mathcal{M})$ and \mathbf{N} of $\pi_1(\mathcal{N})$ and covering of graphs of groups $\mathbf{p} : \mathbf{N} \rightarrow \mathbf{M}$.*

The choice of a maximal tree T in the underlying graph X of a graph of group \mathbf{X} defines embeddings of vertex and edge groups in $\pi_1(\mathbf{X}, x)$. Let $v \in VX$, the embedding $G(v) \rightarrow \pi_1(\mathbf{X}, x)$ is defined by:

$$g \in G(v) \mapsto (1, e_1, 1, \dots, e_n, g, \bar{e}_n, \dots, 1, \bar{e}_1, 1).$$

where (e_1, \dots, e_n) is the simple path in T from x to v . Once embeddings of vertex groups are given, their images in $\pi_1(\mathbf{X}, x)$ are called *vertex subgroups*. Since edge groups embed in vertex groups, embeddings of vertex groups define also embeddings of the edge groups in $\pi_1(\mathbf{X}, x)$; their image in $\pi_1(\mathbf{X}, x)$ are called *edge subgroups* and they all lie in vertex subgroups. For $v \in VX$ and $e \in EX$, the corresponding vertex and edge subgroups will be denoted by G_v, G_e .

Lemma 2.8. *One can construct maximal trees T_N of N and T_M of M such that $\forall e \in T_N, p(e) \in T_M$.*

Proof. Apply a usual algorithm to construct a maximal tree T_N of N : initially T_N is reduced to a vertex of N ; while $VT_N \neq VN$ add to T_N edges e, \bar{e} and vertex v where $v \notin VN \setminus VT_N$ and $t(e) = v, t(\bar{e}) \in VT_N$. Adapt this algorithm to the search of T_M so that $\forall e \in ET_N, p(e) \in ET_M$: initially T_M is reduced to a vertex of M ; while $VT_M \neq VM$ add to T_M edges e, \bar{e} and vertex v where $e \in p(ET_N), v \notin VM \setminus VT_M$ and $t(e) = v, t(\bar{e}) \in VT_M$. It's immediately verified that the algorithm produces a maximal tree T_M with the required property. \square

- *Construct maximal trees T_M of M and T_N of N as above and fix $x \in VM$ and $\tilde{x} \in p^{-1}(x)$; it defines vertex and edges subgroups of $\pi_1(\mathbf{M}, x), \pi_1(\mathbf{N}, \tilde{x})$.*

Now that maximal trees T_M, T_N of M, N and base-points $x \in VM, \tilde{x} \in VN$ are given one can talk of *Seifert vertex subgroups* and *non-Seifert vertex subgroups* of $\pi_1(\mathbf{N}, \tilde{x})$ (respectively as those coming from punctured Seifert fibered pieces, and those which don't) and similarly of $\{1\}$ -edge subgroups, $\mathbb{Z} \oplus \mathbb{Z}$ -edge subgroups, \mathbb{Z}_2 -edge subgroups and $\mathbb{Z} \rtimes \mathbb{Z}$ -edge subgroups of $\pi_1(\mathbf{M}, x)$ and (for the two former) of $\pi_1(\mathbf{N}, \tilde{x})$.

One also partition vertex subgroups of $\pi_1(\mathbf{M}, x)$ into Seifert and non-Seifert vertex subgroups, accordingly to the partition of vertex subgroups in $\pi_1(\mathbf{N}, \tilde{x})$.

In a graph of group \mathbf{X} and given a maximal tree T of X , an (\mathbf{X}, x) -loop γ is said to be *cyclically reduced* whenever:

- (i) γ is a reduced (\mathbf{X}, x) -loop, and
- (ii) either its length is less than 2 or:

$$\gamma = (1, e_1, \dots, 1, e_p, 1)(g_0, e_{p+1}, \dots, e_n, g_n)(1, \bar{e}_p, 1, \dots, \bar{e}_1, 1)$$

where (e_1, \dots, e_p) is the simple path in T from x to $t(e_n)$ (eventually reduced to (x)) and either $e_{p+1} \neq \bar{e}_n$ or $g_n g_0 \notin \phi_{e_n}(G(e_n))$.

We can now state basic algorithms which help working with elements in $\pi_1(\mathbf{M})$ and $\pi_1(\mathbf{N})$.

Lemma 2.9 (Basic algorithms in $\pi_1(\mathbf{M})$). *Let $p : \mathbf{N} \rightarrow \mathbf{M}$ be the covering found above. Fix elements $x \in VM$ and $\tilde{x} \in p^{-1}(x) \subset VN$; then:*

- (i) (Cyclic reduction). *There is an algorithm which given an (\mathbf{M}, x) -loop γ change it into a cyclically reduced (\mathbf{M}, x) -loop γ' and such that γ, γ' represent conjugate elements in $\pi_1(\mathbf{M}, x)$.*
- (ii) (Algorithm $\mathfrak{SWP}(H, G)$). *There is an algorithm which given an (\mathbf{M}, x) -loop γ decides whether γ represents an element of $\pi_1(\mathbf{M}, x)$ which lies in $p_*(\pi_1(\mathbf{N}, \tilde{x}))$, and if so constructs an (\mathbf{N}, \tilde{x}) -loop γ' , with same length then γ , and such that $p_\#(\gamma') = \gamma$. Moreover, whenever γ is reduced (resp. cyclically reduced) then so is γ' .*

Proof. We prove separatly (i) and (ii).

Proof of (i). The first step changes γ into a reduced (\mathbf{M}, x) -loop which represents the same element of $\pi_1(\mathbf{M}, x)$; this is done by application of the generalized word problem for edge subgroups in vertex groups given by Lemma 2.6.(iv). If the reduced (\mathbf{M}, x) -loop obtained, say $\gamma = (g_0, e_1, \dots, e_n, g_n)$ has length $n < 2$, or if $e_1 \neq \bar{e}_n$ then γ is cyclically reduced and the process stops. Otherwise, $n \geq 2$ and $e_1 = \bar{e}_n$; use Lemma 2.6.(iv) to decide whether $g_n g_0 \in \phi_{e_n}(G(e_n))$. If not then the (\mathbf{M}, x) -loop obtained is cyclically reduced and the process stops; if yes change into:

$$\gamma' = (1, e'_1, \dots, 1, e'_p) \underbrace{(\phi_{e_1}(g_n g_0) g_1, e_2, g_2, \dots, e_{n-1}, g_{n-1})}_{\gamma''} (e'_p, 1, \dots, e'_1, 1)$$

where $(1, e'_1, \dots, 1, e'_p)$ is the simple path in T from x to $t(e_1)$; γ'' is a reduced $(\mathbf{M}, t(e_1))$ -loop. Consider the (\mathbf{M}, x) -loop $\alpha = (1, e'_1, \dots, 1, e'_p, 1, \bar{e}_1, g_n)$, then $\gamma' = \alpha \gamma \alpha^{-1}$ in $\pi_1(\mathbf{M}, x)$. If γ'' is cyclically reduced then γ' is cyclically reduced and the process stops. Otherwise apply the

same process to the $(\mathbf{M}, t(e_1))$ -loop γ'' , and so on; after an eventual reduction of the prefix path in T one finally obtains a cyclically reduced (\mathbf{M}, x) -loop which represents a conjugate of γ in $\pi_1(\mathbf{M}, x)$.

Proof of (ii). For each vertex group $H(v)$ of \mathbf{N} one considers its image $p_v(H(v))$ in $G(p(v))$, denoted $H(p(v))$, that one identifies with $H(v)$; it has index at most 2. We consider the generating sets S_v of $G(v)$ and S'_v of $H(v)$ as in Lemma 2.6.(i). For each $v \in VM$, and for each w a word on $S_v \cup S_v^{-1}$ which defines an element in $H(v) \subset G(v)$ denote \bar{w} the word on $S'_v \cup S_v'^{-1}$, which equals w in $G(v)$, given by Lemma 2.6.(ii).

As in the proof of Lemma 2.7, let's denote $VM = VM^+ \sqcup VM^-$ and $VN = \{v^+; v \in EM\} \sqcup \{v^-; v \in EM^+\}$ and elements $\mu(e) \in G(t(p(e)))$ for all $e \in EN$ defined by the covering $\mathbf{p} : \mathbf{N} \rightarrow \mathbf{M}$, and such that $\tilde{x} = t(v^+)$ whenever $x = t(v)$. Given an (\mathbf{M}, x) -loop:

$$\gamma = (g_0, e_1, \dots, e_n, g_n)$$

where $g_i \in G(v_i)$ is given by a word on $S_{v_i} \cup S_{v_i}^{-1}$, one deterministically change into an (\mathbf{N}, \tilde{x}) -loop by the following transformation rules:

$$\begin{aligned} & \forall w_i \text{ word on } S_{t(\bar{e}_{i+1})} \cup S_{t(\bar{e}_{i+1})}^{-1} \\ & w_i, e_{i+1}, w_{i+1} \mapsto \begin{cases} \bar{w}_i, e_{i+1}^+, w_{i+1} & w_i \in H(t(\bar{e}_{i+1})) \\ \bar{w}_i \mu(\bar{e}_{i+1}), e_{i+1}^-, \mu(e_{i+1})^{-1} w_{i+1} & w_i \notin H(t(\bar{e}_{i+1})) \end{cases} \\ & \forall w_n \text{ word on } S_{t(e_n)} \cup S_{t(e_n)}^{-1} \\ & e_n^\pm, w_n \mapsto \begin{cases} e_n^\pm, \bar{w}_n & w_n \in H(x) \\ e_n^\pm, w_n & w_n \notin H(x) \end{cases} \\ & \text{If } n = 0, \forall w_0 \text{ word on } S_x \cup S_x^{-1} \\ & w_0 \mapsto \begin{cases} \bar{w}_0 & w_0 \in H(x) \\ w_0 & w_0 \notin H(x) \end{cases} \end{aligned}$$

If one denotes $\gamma' = (g'_0, e_1^\pm, \dots, e_n^\pm, g'_n)$ the loop obtained, then γ' is an (\mathbf{N}, \tilde{x}) -loop if and only if $g'_n \in H(t(e_n))$ if and only if γ represents an element in $p_*(\pi_1(\mathbf{N}, \tilde{x}))$, and in such case $p_\#(\gamma') = \gamma$. In particular, since $p_\#$ induces an homomorphism $p_* : \pi_1(\mathbf{N}, \tilde{x}) \rightarrow \pi_1(\mathbf{M}, x)$, if γ' is not reduced then neither is γ . The same argument applied in $\pi_1(\mathbf{M}, y)$ for $y = t(e_1)$ shows that if γ is cyclically reduced then so is γ' . \square

• In the following we usually write:

$$G = \pi_1(\mathbf{M}, x) \quad ; \quad H = \pi_1(\mathbf{N}, \tilde{x})$$

and see H as a subgroup of G by mean of the monomorphism $p_* : H \rightarrow G$.

We recapitulate what one knows concerning basic Dehn problems in G and H .

Lemma 2.10. *There exists algorithms which solve the following problems in G and H :*

- (i) (Algorithms $\mathfrak{WP}(H)$, $\mathfrak{CP}(H)$): *the word and conjugacy problems in H .*
- (ii) (Algorithm $\mathfrak{WP}(G)$): *the word problem in G .*
- (iii) (Algorithms $\mathfrak{WP}(H(v))$, $\mathfrak{CP}(H(v))$): *the word and conjugacy problems in vertex subgroups of H .*
- (iv) (Algorithms $\mathfrak{WP}(G(v))$, $\mathfrak{CP}(G(v))$): *the word and conjugacy problems in vertex subgroups of G .*

Proof. Algorithms solving (i), (ii) are constructed respectively in [Pr] and [CEHLPT]. Let H_v be a vertex subgroup of H ; in such case the solution is provided by [CEHLPT, NR] and [Pr] depending on whether H_v is biautomatic or the piece is modelled on *Nil*-geometry. This proves (iii). Let G_v be a vertex subgroup of G ; in cases it comes from an orientable piece apply the same process as in (iii). Otherwise: a non-orientable 3-manifold cannot be modelled on *Nil*-geometry (cf.[Sc]); it follows from [CEHLPT, NR] that G_v is biautomatic which allows to solve word and conjugacy problems in G_v . This proves (iv). \square

We will make a heavy use of these basic algorithms. Another of the main ingredients will be the following algorithm which finds the centralizers of elements in H . In case of Haken orientable 3-manifolds the structure of centralizers are quite simple and related to the JSJ decomposition as stated in Theorem VI.I.6 [JS]; one deduces centralizers in groups of geometrizable orientable 3-manifolds, that one can compute, as follows.

Lemma 2.11 (Algorithm $\mathfrak{ZP}(H)$ for centralizers in H). *Let $u \in H \setminus \{1\}$ as above; then exactly one of the following assertions occurs:*

- (i) *its centralizer $Z_H(u)$ is infinite cyclic and is not in the conjugate of a Seifert vertex subgroup,*
- (ii) *$Z_H(u)$ lies in the conjugate of a Seifert vertex subgroup H_v ,*
- (iii) *$Z_H(u)$ is conjugate to a $\mathbb{Z} \oplus \mathbb{Z}$ edge subgroup H_e and is not in the conjugate of a Seifert vertex subgroup.*

There is an algorithm which determines which case occurs, and in cases (ii) and (iii) produces all possible vertex or edge subgroups H_v , H_e and conjugating elements.

Proof. First note that in case where the tori decomposition of \mathcal{N} is empty, then, since in groups of hyperbolic closed manifolds non trivial centralizers are all infinite cyclic (cf. [Sc]), the former assumption follows from Theorem VI.I.6 and the latter assumption from Lemma 2.5. So we suppose in the following that the tori decomposition of \mathcal{N} is non-empty.

We are given an element $u \in H$ by a (\mathbf{N}, \tilde{x}) -loop and are interested in its centralizer; first apply algorithm in Lemma 2.9.(i) to $p_{\#}(u)$ followed by algorithm in Lemma 2.9.(ii) to change u into a cyclically reduced (\mathbf{N}, \tilde{x}) -loop v conjugate to u in H ; it finds $h \in H$ such that $huh^{-1} = v$.

Obviously $Z_H(v) = h Z_H(u) h^{-1}$, so that in the following we suppose that u is a cyclically reduced (\mathbf{N}, \tilde{x}) -loop.

Let EN_0 be the subset of EN of those edges whose groups are all trivial (*i.e.* coming from \mathbf{S}^2 in the topological decomposition of \mathcal{N}). Denote N_1, \dots, N_p the connected components of the graph obtained from N by deleting edges in EN_0 . Together with \mathbf{N} it defines graphs of groups $\mathbf{N}_1, \dots, \mathbf{N}_p$ by restricting \mathbf{N} to the respective subgraphs N_1, \dots, N_p . The choice of base points $\tilde{x}_1, \dots, \tilde{x}_p$ in N_1, \dots, N_p together with the maximal tree T_N defines natural embeddings of $\pi_1(\mathbf{N}_1, \tilde{x}_1), \dots, \pi_1(\mathbf{N}_p, \tilde{x}_p)$ into $H = \pi_1(\mathbf{N}, \tilde{x})$. Moreover H splits as the free product $\pi_1(\mathbf{N}, \tilde{x}_1) * \dots * \pi_1(\mathbf{N}_p, \tilde{x}_p) * F_n$, for F_n a free group of finite rank. Note that $\mathbf{N}_1, \dots, \mathbf{N}_p$ are graphs of groups related to the decompositions of the $\hat{\mathcal{N}}_1, \dots, \hat{\mathcal{N}}_p$ along tori of Lemma 2.4, and that embeddings of $\pi_1(\mathbf{N}_1, \tilde{x}_1), \dots, \pi_1(\mathbf{N}_p, \tilde{x}_p)$ into $H = \pi_1(\mathbf{N}, \tilde{x})$ coincide up to conjugacy with those induced by the inclusions of $\mathcal{N}_1, \dots, \mathcal{N}_p$ in \mathcal{N} .

From u one obtains readily a cyclically reduced sequence according to the free products. If it has length > 0 then $Z_H(u)$ is infinite cyclic, case (i) occurs, and otherwise $Z_H(u)$ lies in one of the free products factors (corollary 4.1.6, [MKS]). In the latter case it follows from Theorem VI.1.6 and the Characteristic Pair Theorem in [JS] that exactly one of the assertions (i), (ii), (iii) occurs; this proves the first assumption.

We now return to the algorithm: if u passes through an edge in EN_0 , then case (i) occurs otherwise $Z_H(u)$ is included in, say, $\pi_1(\mathbf{N}_1, \tilde{x}_1)$ that one can determine readily from u . Suppose in the following that u lies in $K \triangleq \pi_1(\mathbf{N}_1, \tilde{x}_1)$, and write u readily as a cyclically reduced $(\mathbf{N}_1, \tilde{x}_1)$ -loop.

The algorithm is constructed on procedures and arguments stated in [Pr] which applies here since \mathbf{N}_1 is related to the tori decomposition of the orientable irreducible geometrizable 3-manifold $\hat{\mathcal{N}}_1$. Suppose that u is conjugate in K to an element u' in a vertex subgroup $H_{v'}$, for $v' \in \text{VN}_1$. According to Theorem 3.1 in [Pr], since u is cyclically reduced one of the following cases occurs in $K = \pi_1(\mathbf{N}_1, \tilde{x}_1)$:

- (i) u lies in $H_{v'}$ and u, u' are conjugate in $H_{v'}$, or
- (ii) u lies in a vertex subgroup H_v , $v \in \text{VN}_1$, and there is a sequence (c_1, \dots, c_n) of elements of edge subgroups such that u is conjugate to c_1 in H_v , u' is conjugate to c_n in $H_{v'}$ and for $i = 1 \dots n - 1$, either $(e, c_i, \bar{e}) = c_{i+1}$ for some $e \in \text{EN}_1$ or c_i, c_{i+1} are conjugate in a vertex subgroup.

Let's return to the algorithm; given u a cyclically reduced $(\mathbf{N}_1, \tilde{x}_1)$ -loop one can decide readily whether u lies in a vertex subgroup. If not then with the above u is not conjugate to a vertex subgroup and assertion (i) occurs: $Z_H(u)$ is infinite cyclic. So in the following we will suppose that u lies in a vertex subgroup H_v of K , for some $v \in \text{EN}_1$.

First consider the particular case where $\hat{\mathcal{N}}_1$ is a \mathbf{T}^2 -bundle over \mathbf{S}^1 modelled on Sol (cf. Theorem 5.3, [Sc]). This occurs when N_1 is a cycle with one or two vertices (resp. edges) and all vertex and edge groups are free abelian with rank 2. In such case K splits as $(\mathbb{Z} \oplus \mathbb{Z}) \rtimes_{\theta} \mathbb{Z}$ for some $\theta \in SL_2\mathbb{Z}$ anosov and the left factor coincide with all vertex and edge subgroups of K (while embeddings of edge groups are not in general equal). It follows easily from the fact that θ has no eigenval with modulus 1 that the centralizer of any element in K is either infinite cyclic or consists in the left factor $\mathbb{Z} \oplus \mathbb{Z}$. Hence here assertion (ii) occurs.

Now consider the remaining cases where $\hat{\mathcal{N}}_1$ is not a \mathbf{T}^2 -bundle over \mathbf{S}^1 modelled on Sol . Using the Seifert invariants obtained by algorithm **Top4** (Lemma 2.5) one decides which vertex subgroup is a Seifert subgroup and among them which comes from a $\mathbf{T}^2 \times \mathbf{I}$ piece (those with basis an annulus and no exceptionnal fiber); note that the latter correspond to vertex subgroups which are free abelian with rank 2 (Theorem 10.5, [He]). Use the following process to find all elements in vertex subgroups conjugate to u in H_v :

- If H_v is not a Seifert vertex subgroup; then according to proposition 4.1 [Pr], u is conjugate in H_v to at most 1 element lying in at most one edge subgroup that, using Theorem 6.3 of [Pr], one determines as well as a conjugating element in H_v .
- If $H_v \simeq \mathbb{Z} \oplus \mathbb{Z}$; u lies in the two edge subgroups and is not conjugate in H_v to any other element.
- If H_v is a Seifert vertex subgroup and $H_v \not\simeq \mathbb{Z} \oplus \mathbb{Z}$. According to proposition 4.1 [Pr], either u is conjugate in H_v to at most 1 element lying in at most one edge subgroup, or u lies in a fiber of H_v , *i.e.* is a power of a regular fiber in a Seifert fibration of the corresponding piece and u lies in the intersection of all edge subgroups in H_v . One decides using Proposition 5.1 of [Pr] (note also that deciding whether u lies in a fiber of H_v is easily done by checking with a solution to the word problem whether for all generators s of H_v , $sus^{-1} = u^{\pm 1}$, cf. Lemma II.4.2.(i) [JS]).

Pursue the process with the successive conjugates in the edge subgroups obtained, the acylindricity of \mathbf{N}_1 (Lemma 4.1, [Pr]) ensures that it finally stops and one finally obtains a finite list of all elements in vertex and edge subgroups to which u is conjugate, as well as conjugating elements.

The minimality of the JSJ decomposition of $\hat{\mathcal{N}}_1$ ensures that u is not conjugate to the fibers of two Seifert vertex subgroups $\not\simeq \mathbb{Z} \oplus \mathbb{Z}$, neither to the fibers of two Seifert vertex subgroups $\simeq \mathbb{Z} \oplus \mathbb{Z}$. Then with the above, together with Theorem VI.I.6 of [JS], one finally obtains exactly one of the following cases:

- u is conjugate neither to a Seifert vertex subgroup nor to an edge subgroup: $Z_H(u) \simeq \mathbb{Z}$ and assertion (i) occurs.

- u is conjugate to a Seifert vertex subgroup $H_{v'}$ and is not conjugate to a fiber of any Seifert vertex subgroup $\not\cong \mathbb{Z} \oplus \mathbb{Z}$. In that case $Z_H(u)$ lies up to conjugacy in $H_{v'}$ and assertion (ii) occurs.
- u is conjugate to the fiber of a Seifert vertex subgroup $H_{v'} \not\cong \mathbb{Z} \oplus \mathbb{Z}$; $Z_H(u)$ lies up to conjugacy in $H_{v'}$ and assertion (ii) occurs.
- u is not in the conjugate of any Seifert vertex subgroup but lies in the conjugate of an edge subgroup H_e ; $Z_H(u) = H_e$ and assertion (iii) occurs.

This achieves the proof. \square

2.4. Step 4: The conjugacy algorithm. We construct in this section the algorithm solving conjugacy problem in G . When for u, v, h lying in a group $u = hvh^{-1}$ we shall use the notation $u = v^h$ or $u \sim v$.

- Suppose u and $v \in G$ are given by a couple of (\mathbf{M}, \mathbf{x}) -loops and one wants to decide whether $u \sim v$ in G .

First use the solution $\mathfrak{SWP}(H, G)$ (Lemma 2.9.(ii)) to the generalized word problem of H in G to decide whether u, v lie in H other not.

- If either u or v lies in H .

Since H has index 2 in G , if u and v lie in distinct classes of H/G they are definitely not conjugate in G . If u and v both lie in H , then the solution $\mathfrak{CP}(H)$ (Lemma 2.10.(i)) to the conjugacy problem in H together with the following lemma allow to decide whether u and v are conjugate in G .

Lemma 2.12 (Algorithm $\mathfrak{CP}_1(K)$). *Let K be a group and L an index 2 subgroup of K with solvable conjugacy problem. Given any couple of elements $u, v \in L$ one can decide whether u and v are conjugate in K .*

Proof. Given a set of representative $a_0 = 1, a_1$ of L/K , in order to decide whether $u, v \in L$ are conjugate in K it suffices to decide whether u is conjugate in L to any of the $a_i v a_i^{-1}$ for $i = 0, 1$. \square

- In the following both u and v lie in $G \setminus H$ and are supposed cyclically reduced.

For that apply the algorithm in Lemma 2.9.(i) to change u and v into two cyclically reduced (\mathbf{M}, \mathbf{x}) -loops, respective conjugates of u, v in G .

Decide whether u, v have order 2; according to [Se] it occurs when u, v lie in vertex subgroups of G , so use a solution to the word problem in those vertex subgroups (Lemma 2.10.(iv)) to check whether $u^2 = 1$,

$v^2 = 1$ (or $\mathfrak{WP}(G)$, Lemma 2.10.(ii)). If exactly one of the relations occurs then u and v are not conjugate in G .

- *If both u and v have order 2.*

In such case the following lemma allows to decide whether u and v are conjugate or not.

Lemma 2.13 (Algorithm $\mathfrak{EP}_2(G)$). *One can decide for any pair of order 2 elements $u, v \in G$ whether u and v are conjugate in G .*

Proof. Recall the system \mathcal{P} of essential projective planes in \mathcal{M} as in §2.2 ; they are necessarily pairwise non parallel. It follows from [Ep], [St], [Sw] that each order 2 element in G is conjugate to some \mathbb{Z}_2 -edge subgroup of G and that all \mathbb{Z}_2 -edge subgroups are pairwise non conjugate in G .

Let u, v be cyclically reduced element of order 2 lying in respective vertex subgroups $G_v, G_{v'}$. According to [Ep] (Theorem 9.8.(i), [He]) and Proposition 2.2, [Sw], u and v are necessarily conjugate in $G_v, G_{v'}$ to the generators of \mathbb{Z}_2 -edge subgroups. One decides so using the solutions $\mathfrak{EP}(G_v), \mathfrak{EP}(G_{v'})$ (Lemma 2.10.(iv)) to the conjugacy problems in G_v and $G_{v'}$. Then u, v are conjugate in G if and only if they are conjugate to the non-trivial element in a \mathbb{Z}_2 -vertex subgroup, or to two non-trivial elements in two \mathbb{Z}_2 -vertex subgroups coming from opposite edges $e, \bar{e} \in \text{EM}$. \square

We will be concerned in the following with the remaining case: u, v both lie in $G \setminus H$ and both have order different than 2. According to [Ep] both u and v must have infinite order in G .

- *In the following both u and v lie in $G \setminus H$ and have infinite order.*

Use algorithm $\mathfrak{EP}_1(G)$ (Lemma 2.12) to decide whether $u^2 \sim v^2$ in G and find if any $k \in G$ that conjugates u^2 into v^2 ; if such $k \in G$ does not exist then u, v are not conjugate in G . So we suppose in the following that $u^2 \sim v^2$ in G and we are given an element $k \in G$ such that $u^2 = (v^2)^k$ in G ; such a conjugating element is implicitly provided (going into the lines of the proof) by the conjugacy algorithm in [Pr].

- *In the following u^2 and v^2 are conjugate in G and we are given $k \in G$ such that $u^2 = (v^2)^k$.*

We first need to fix some notations which will be useful in the following. Let denote $Z_G(v) = \{u \in G \mid uv = vu\}$ the centralizer of v in G and $C_G(u, v) = \{k \in G \mid u = v^k\}$. The subset $C_G(u, v)$ of G is either empty (when $u \not\sim v$) or $k \cdot Z_G(v)$ for any $k \in G$ such that $u = v^k$. The set $C_G(u^2, v^2) = k \cdot Z_G(v^2)$ is non empty. It obviously contains the

set $C_G(u, v)$; note also that $Z_G(v^2)$ contains $Z_G(v)$ as a subgroup as well as $Z_H(v^2)$ as an index 2 subgroup; $Z_G(v^2)$ is generated by $Z_H(v^2)$ and v .

We are now interested in the centralizer $Z_H(v^2)$ of v^2 in H . Apply algorithm $\mathfrak{3P}(H)$ (Lemma 2.11) to check whether it is infinite cyclic or conjugate into a Seifert vertex subgroup or to a $\mathbb{Z} \oplus \mathbb{Z}$ -edge subgroup of H . It provides the eventual Seifert vertex subgroup H_v or edge subgroup H_e of H and conjugating element $h \in H$ such that $hZ_H(v^2)h^{-1}$ lies in H_v or H_e .

- Check whether $Z_H(v^2)$ is infinite cyclic or is conjugate to a $\mathbb{Z} \oplus \mathbb{Z}$ -edge subgroup or into a Seifert vertex subgroup of H .

We now treat separately the former case and the two latter cases.

- Case (i): $Z_H(v^2)$ is infinite cyclic.

In the case where $Z_H(v^2)$ is infinite cyclic, $Z_G(v^2)$ contains \mathbb{Z} as an index 2 subgroup. If $Z_G(v^2)$ is torsion-free then it must be cyclic, say $Z_G(v^2) = \langle w \rangle$. But since $v \in Z_G(v^2)$, v is a power of w so that $w \in Z_G(v)$, and since $Z_G(v) \subset Z_G(v^2)$, it implies that $Z_G(v) = Z_G(v^2) = \langle w \rangle$. If $Z_G(v^2)$ has torsion, let us denote by t a generator of its index 2 subgroup $Z_H(v^2)$. The group $Z_G(v^2)$ is generated by v and t and must be one of the two groups appearing in the following lemma.

Lemma 2.14 (Groups with torsion containing \mathbb{Z} as an index 2 subgroup). *A group K with torsion and generators v, t , such that $\langle t \rangle \simeq \mathbb{Z}$ has index 2 in K must be one of:*

$$\langle v, t \mid [v, t] = 1, v^2 = t^{2n} \rangle \simeq \mathbb{Z} \oplus \mathbb{Z}_2$$

$$\langle v, t \mid t^v = t^{-1}, v^2 = 1 \rangle \simeq \mathbb{Z}_2 * \mathbb{Z}_2$$

Proof. The group K admits the presentation $\langle v, t \mid t^v = t^{\pm 1}, v^2 = t^p \rangle$ for some $p \in \mathbb{Z}$. The set $K \setminus \langle t \rangle$ contains an element w with finite order $m \neq 0$. In particular w^m lies in the index 2 subgroup $\langle t \rangle$ so that m must be even; hence K contains an element $t^{-n}v$ with order 2, for some $n \in \mathbb{Z}$. Suppose first that $t^v = t$, so that $1 = (t^{-n}v)^2 = t^{-2n+p}$. It follows that $p = 2n$, which gives the first presentation. Suppose then that $t^v = t^{-1}$; one has $1 = (t^{-n}v)^2 = v^2$ which gives the second presentation. \square

The latter group cannot occur since v has infinite order. Concerning the former group, since $[v, t] = 1$, one has $Z_G(v) = Z_G(v^2)$. Hence whenever $Z_H(v^2)$ is infinite cyclic then $Z_G(v) = Z_G(v^2)$ and the following lemma allows us to decide whether $u \sim v$ in G .

Lemma 2.15 (Algorithm $\mathfrak{CP}_3(K)$). *Let K be a group and L be an index 2 subgroup of K . Suppose that L has solvable conjugacy problem.*

Let $v \in K \setminus L$ such that $Z_K(v) = Z_K(v^2)$. Then one can decide for any $u \in K$ whether u and v are conjugate in K .

Proof. Since L has solvable conjugacy problem, L has solvable word problem, and hence K also has solvable word problem. Let $v \in K$ be as above, and suppose one wants to decide for some given $u \in K$ whether $u \sim v$ in K . With lemma 2.12 one can decide whether u^2 and v^2 are conjugate in K . If not then u and v are definitely not conjugate in K . So suppose that $u^2 = kv^2k^{-1}$ for some $k \in K$ that one can effectively find using a solution to the word problem in K (in our purpose k is provided by the solution of [Pr] to conjugacy in H), so that $C_K(u^2, v^2) = kZ_K(v^2)$. Obviously $C_K(u, v) \subset C_K(u^2, v^2)$ and moreover since $Z_K(v^2) = Z_K(v)$, if $C_K(u, v)$ is non empty it must equal $C_K(u^2, v^2)$. Hence to decide whether u and v are conjugate in K it suffices to decide with the word problem in K whether $u = v^k$ or not. \square

- Case (ii) and (iii): $Z_H(v^2)$ is conjugate in H to a subgroup of a Seifert vertex subgroup or to a $\mathbb{Z} \oplus \mathbb{Z}$ -edge subgroup of H .

Let $h \in H$ be given by algorithm $\mathfrak{3P}(H)$ (Lemma 2.11) such that $hZ_H(v^2)h^{-1}$ lies into a Seifert vertex subgroup H_v or an edge subgroup H_e of H . Let $G_v \supset H_v$ and $G_e \supset H_e$ be the corresponding Seifert vertex subgroup or edge subgroup in G . Since $hZ_H(v^2)h^{-1}$, and in particular hv^2h^{-1} , lie in G_v or G_e one may expect that hvh^{-1} also lies in G_v , G_e . This turns to be false in general, though only in specific cases.

Lemma 2.16 (Square root of an element of G lying in a vertex or edge subgroup). *Let v be a cyclically reduced element of $G \setminus H$ with infinite order and $h \in H$, such that $Z_H(v^2)^h$ lies in a vertex or edge subgroup of G . Then either:*

- (i) $Z_H(v^2)^h$ and v^h both lie in a same vertex subgroup or edge subgroup of G , or
- (ii) v lies in a vertex subgroup which contains $\mathbb{Z} \oplus \mathbb{Z}$ as an index 2 subgroup, and do not lie in any edge subgroup.

Proof. Consider the cyclically reduced (\mathbf{M}, \mathbf{x}) -loop v , it takes one of the following forms:

$$v = (1, e_1, 1, \dots, e_n, 1)(v_n, e_{n+1}, \dots, e_m, v_m)(1, \bar{e}_n, \dots, 1, \bar{e}_1, 1)$$

or $v = (v_1, e, v_2)$, or $v = (v_0)$. Then v^2 takes one of the forms:

$$v^2 = (1, e_1, 1, \dots, e_n, 1)(v_n, e_{n+1}, \dots, e_m, v_m v_n, e_{n+1}, \dots, e_m, v_m) \\ (1, \bar{e}_n, \dots, 1, \bar{e}_1, 1)$$

or $v^2 = (v_1, e, v_2 v_1, e, v_2)$, or $v^2 = (v_0^2)$, which are cyclically reduced except possibly in the first case when $m = n$. Hence if v^2 lies in vertex (or edge) subgroup, then v also lies in a vertex subgroup (not necessarily the same) of G .

Write, using algorithm of Lemma 2.9.(ii), v^2 into a cyclically reduced $(\mathbf{N}, \tilde{\mathbf{x}})$ -loop; v^2 is conjugate to a vertex or edge subgroup of H and according to Theorem 3.1 in [Pr] v^2 lies in a vertex subgroup H_v of H and is conjugate in H_v to one of its edge subgroup H_e . Hence v, v^2 both lie in a vertex subgroup G_v of G and v^2 is conjugate in G_v to an edge subgroup G_e . Moreover if any time that v, v^2 both lie in a conjugate wG_vw^{-1} of a vertex subgroup and zv^2z^{-1} , for some $z \in wG_vw^{-1}$, lies in $wG_e w^{-1}$, for an edge subgroup G_e of G_v , one has that $zvvz^{-1}$ also lie in $wG_e w^{-1}$, then $\forall g \in G$, whenever gv^2g^{-1} lies in a vertex or edge subgroup of G , gvg^{-1} lies in the same vertex or edge subgroup. In particular assumption (i) occurs.

So suppose in the following that, up to conjugacy, v, v^2 both lie in a vertex subgroup G_v , and for some $z \in H_v$, zv^2z^{-1} lies in an edge subgroup G_e of G_v while $zvvz^{-1}$ does not. Since $v \notin H$, the piece \mathcal{M}_v in the topological decomposition of \mathcal{M} associated to the vertex v is non-orientable; let \mathcal{N}_v be its orientation cover. Since v has infinite order the edge e can be associated to either \mathbf{T}^2 or \mathbf{K}^2 .

First case: e is associated to \mathbf{T}^2 . Then \mathcal{N}_v has two \mathbf{T}^2 in its boundary which give rise up to conjugacy to two edge subgroups G_e and $vG_e v^{-1}$ in H_v . But since $zv^2z^{-1} \in G_e$, v^2 both lies in conjugates of G_e and $vG_e v^{-1}$ in H_v (with respective conjugating elements z^{-1} and $vz^{-1}v^{-1}$). Hence by Proposition 4.1 of [Pr], \mathcal{N}_v is a Seifert fiber space and v^2 is a power of a regular fiber in a Seifert fibration. If $H_v \not\cong \mathbb{Z} \oplus \mathbb{Z}$ then (see the end in the proof of Lemma 2.11) $Z_H(v^2) \subset H_v$ and $Z_G(v^2) \subset G_v$; assumption (i) occurs. Otherwise v lies in a vertex subgroup containing $\mathbb{Z} \oplus \mathbb{Z}$ as an index 2 subgroup.

Second case: e is associated to \mathbf{K}^2 . Then \mathcal{N}_v has one \mathbf{T}^2 in its boundary which gives rise to the vertex subgroup $H_e = G_e \cap H$ of H . Since $zv^2z^{-1} \in H_e$, one has that v^2 both lie in conjugates of H_e and $vH_e v^{-1}$ in H_v (with respective conjugating elements z^{-1} and $vz^{-1}v^{-1}$). Let $w \in G_v$ such that H_e and w generate G_e ; there exists $t \in H_v \setminus H_e$ such that $v = tw$. Since $wH_e w^{-1} = H_e$ one has also that v^2 lies in two conjugates of H_e in H_v (with respective conjugating elements z^{-1} and $vz^{-1}v^{-1}t$). Since $zvvz^{-1} \notin G_e$, one has $zvvz^{-1}v^{-1}t \notin H_e$ and it follows from Proposition 4.1 of [Pr] that v^2 is a power of a regular fiber in a Seifert fibration of $\hat{\mathcal{N}}_v$. The conclusion follows as in the first case.

By taking successive conjugates of v in adjacent vertex subgroups of G , one finally obtains that either assumption (i) occurs or at some stage one obtains a conjugate of v which lies in a vertex subgroup G_v containing $\mathbb{Z} \oplus \mathbb{Z}$ as an index 2 subgroup, and does not lie in some conjugate in G_v of an edge subgroup G_e containing $\mathbb{Z} \oplus \mathbb{Z}$. Note using the following Lemma, that in such case, since on the one hand such edge subgroup is normal and on the other v has infinite order, v is not conjugate in G_v to any of the edge subgroups of G_v . So finally, in case one of the successive conjugates of v lies in such a vertex subgroup G_v

containing a $\mathbb{Z} \oplus \mathbb{Z}$ of index 2, then this can arise only at initial step, that is $v \in G_v$: in such case conclusion (ii) occurs. \square

Lemma 2.17 (Description of the non-orientable pieces whose groups contain $\mathbb{Z} \oplus \mathbb{Z}$ as an index 2 subgroup). *Let G_v be a vertex subgroup of G which is not included in H and which contains $\mathbb{Z} \oplus \mathbb{Z}$ as an index 2 subgroup. Let \mathcal{M}_v be the 3-manifold associated to the vertex v of \mathbf{M} and let $St(v) = \{e \in EM, t(e) = v\}$. Then exactly one of the following cases occurs:*

- (1) \mathcal{M}_v is the product of a Moebius band and \mathbf{S}^1 , $\partial\mathcal{M}_v$ consists of one \mathbf{T}^2 , $St(v) = \{e\}$,

$$G_v = \langle a, b, t \mid [a, b] = [b, t] = 1, t^2 = a \rangle \simeq \mathbb{Z} \oplus \mathbb{Z}$$

and $G_e \simeq \mathbb{Z} \oplus \mathbb{Z}$ is generated by a and b .

- (2) \mathcal{M}_v is homeomorphic to $\mathbf{K}^2 \times \mathbf{I}$, $\partial\mathcal{M}_v$ consists of two \mathbf{K}^2 , $St(v) = \{e_1, e_2\}$,

$$G_e = \langle a, b, t \mid [a, b] = 1, b^t = b^{-1}, t^2 = a \rangle \simeq \mathbb{Z} \rtimes \mathbb{Z}$$

and $G_{e_1} = G_{e_2} = G_v$.

- (3) $\partial\mathcal{M}_v$ consists of one \mathbf{T}^2 and four \mathbf{P}^2 ; $St(v) = \{e_0, e_1, e_2, e_3, e_4\}$, $\langle a, b, t \mid [a, b] = 1, a^t = a^{-1}, b^t = b^{-1}, t^2 = 1 \rangle \simeq (\mathbb{Z} \oplus \mathbb{Z}) \rtimes_{-I} \mathbb{Z}_2$

$G_{e_0} = \langle a, b \rangle \simeq \mathbb{Z} \oplus \mathbb{Z}$ and $G_{e_i} \simeq \mathbb{Z}_2$, $i = 1 \dots 4$, are generated respectively by: t , at , bt , abt .

Moreover, two elements $u = a^{n_1}b^{m_1}t$ and $v = a^{n_2}b^{m_2}t$ are conjugate in G_v if and only if, respectively:

$$(1) \begin{cases} n_1 = n_2 \\ m_1 = m_2 \end{cases} ; \quad (2) \begin{cases} n_1 = n_2 \\ m_1 = m_2 \pmod{2} \end{cases} ; \quad (3) \begin{cases} n_1 = n_2 \pmod{2} \\ m_1 = m_2 \pmod{2} \end{cases}$$

Proof. The group G_v is the fundamental group of the non-orientable 3-manifolds \mathcal{M}_v which is two covered by a possibly punctured $\mathbf{T}^2 \times \mathbf{I}$; the cover involution extends to an orientation reversing involution σ of $\mathbf{T}^2 \times \mathbf{I}$ with at most isolated fixed points. It is known (cf. details in [LS]) that there are up to isotopy 5 involutions with at most isolated fixed points on $\mathbf{T}^2 \times \mathbf{I}$, among which 3 are non-orientable. We set here $I = [0, 1]$ and $\mathbf{S}^1 = \mathbb{R}/2\pi\mathbb{Z}$. According to [KT], up to isotopy σ factors as a product $\sigma((x, y), t) = (\phi(x, y), t)$ or $\sigma((x, y), t) = (\phi(x, y), 1 - t)$ for ϕ an homeomorphism of \mathbf{T}^2 . There are up to isotopy 5 involutions on the torus. They are:

1. $\phi(x, y) = (x + \pi, y)$ with null fixed point set and orbit space \mathbf{T}^2 ,
2. $\phi(x, y) = (-x, y)$ with fixed point set $\mathbf{S}^1 \times \mathbf{S}^0$ and orbit space $\mathbf{S}^1 \times I$,
3. $\phi(x, y) = (y, x)$ with fixed point set a circle and orbit space a Moebius band,
4. $\phi(x, y) = (x + \pi, -y)$ with null fixed point set and orbit space \mathbf{K}^2 ,
5. $\phi(x, y) = (-x, -y)$ with fixed point set 4 points and orbit space \mathbf{S}^2 ,

among them only 1 and 5 are orientation preserving. Since σ is non-orientable and has at most isolated fixed points, the only 3 possibilities are:

- (1). $\sigma(x, y, t) = (x + \pi, y, 1 - t)$; $\mathcal{M}_v = \mathbf{T}^2 \times \mathbf{I}/\sigma$ is the twisted I -bundle over the torus, otherwise said the product of a Moebius band and \mathbf{S}^1 , and G_v admits the first presentation.
- (2). $\sigma(x, y, t) = (x + \pi, -y, t)$; $\mathcal{M}_v = \mathbf{T}^2 \times \mathbf{I}/\sigma$ is $\mathbf{K}^2 \times \mathbf{I}$ and G_v admits the second presentation.
- (3). $\sigma(x, y, t) = (-x, -y, 1 - t)$; $\mathcal{M}_v = \mathbf{T}^2 \times \mathbf{I}/\sigma$ has orientation cover $\mathbf{T}^2 \times \mathbf{I}$ minus 4 balls centered on the fixed points, its boundary consists of four \mathbf{P}^2 and one \mathbf{T}^2 , and G_v admits the third presentation. One can see \mathcal{M}_v in the following way: call \mathbf{P} two copies of $\mathbf{P}^2 \times \mathbf{I}$ glued along two disks in their boundary, $\partial\mathbf{P}$ consists of one \mathbf{K}^2 and two \mathbf{P}^2 and the \mathbf{K}^2 contains one annulus wich is essential in \mathbf{P} . Glue two copies of \mathbf{P} on this annulus to obtain \mathcal{M}_v (cf. details in [LS]).

The conjugacy criteria are obtained by direct computation. \square

Now several cases can occur, that one decides using the following lemma.

Lemma 2.18. *One of the following cases occur:*

- (a) $Z_G(v^2)$ is conjugate to a subgroup of a Seifert vertex subgroup G_v of G .
- (b) $Z_G(v^2)$ is conjugate to an edge subgroup G_e of G .
- (c) $Z_G(v^2)$ is conjugate neither in a Seifert vertex subgroup nor to an edge subgroup of G . Both u and v lie in vertex subgroups containing $\mathbb{Z} \oplus \mathbb{Z}$ as an index 2 subgroup, and do not lie in any edge subgroup.

One can decide which case occurs.

Proof. That cases (a), (b) or (c) occurs follows from Lemma 2.16 applied to u and v (when applying Lemma 2.16 to u , if $k \notin h$ use conjugating element $hk^{-1}u \in H$ rather than $hk^{-1} \notin H$), since $Z_G(v^2)$ is generated by on the one hand $Z_H(v^2)$ and v and on the other by $Z_H(v^2)$ and u . The algorithm $\mathfrak{ZP}(H)$ returns all vertex or edge subgroups of H containing $Z_H(v^2)$. Using algorithm in Lemma 2.6.(iv) one finds all vertex and edge subgroups containing v , u . A vertex subgroup G_v contains $\mathbb{Z} \oplus \mathbb{Z}$ as an index 2 subgroup if and only if H_v is associated to a Seifert piece with among its Seifert invariants, has basis $\mathbf{S}^1 \times \mathbf{I}$ and no exceptional fiber. \square

- Case (a) and (b): $Z_G(v^2)$ is conjugate to a subgroup of a Seifert vertex subgroup G_v or to an edge subgroup G_e of G .

In those cases (a) and (b) change v into hvh^{-1} and u into $hkh^{-1}ukh^{-1}$ so that u, v both lie in G_v or G_e and $u^2 = v^2$.

- Change v and u into their respective conjugate hvh^{-1} and $hk^{-1}ukh^{-1}$ in G_v, G_e .

We now return to each of the cases (a), (b).

- Case (a): $Z_G(v^2)$ lies in a Seifert vertex subgroup G_v of G .

One decides whether $u \sim v$ in G , using the following lemma and the solution to the conjugacy problem in G_v (Lemma 2.10.(iv)).

Lemma 2.19 (In case $Z_G(v^2)$ is included in a Seifert vertex group). *If $Z_G(v^2)$ is included in a Seifert subgroup G_v of G , then a solution to the conjugacy problem in G_v allows to decide whether $u \sim v$ in G .*

Proof. One has by hypothesis that $C_G(u^2, v^2) = Z_G(v^2)$ is included in G_v . Since $C_G(u^2, v^2) \supset C_G(u, v)$, if u and v are conjugate in G , they must also be conjugate in G_v , that one can decide using $\mathfrak{CP}(G_v)$ (Lemma 2.10.(iv)). \square

- Case (b): $Z_G(v^2)$ lies in an edge subgroup G_e of G .

In such case one decides whether $u \sim v$ using the following lemma.

Lemma 2.20 (Algorithm $\mathfrak{CP}(\mathbb{Z} \rtimes \mathbb{Z})$). *If $Z_G(v^2)$ is included in an edge subgroup G_e of G , then one can decide whether $u \sim v$ in G .*

Proof. Necessarily $G_e = Z_G(v^2)$ is isomorphic to the group of the Klein bottle which has generators t, b in G with finite presentation $G_e = \langle t, b \mid tbt^{-1} = b^{-1} \rangle$. Set $a = t^2$ to obtain the alternate presentation $\langle a, b, t \mid [a, b] = 1, t^2 = a, b^t = b^{-1} \rangle$; then use the algorithm in Lemma 2.6.(iv) to write u and v on generators a, b, t , say $u = a^{n_1}b^{m_1}t$ and $v = a^{n_2}b^{m_2}t$. Then (Lemma 2.17) $u \sim v$ in G_v if and only if $n_1 = n_2$ and $m_1 = m_2 \pmod{2}$, and one decides whether $u \sim v$ in G_e . Since here again $C_G(u, v) = Z_G(v) \subset G_e$, one has finally $u \sim v$ in G_e if and only if $u \sim v$ in G . \square

- Case (c): $Z_G(v^2)$ is conjugate neither to a subgroup of a vertex subgroup nor to an edge subgroup of G .

In that case both u and v lie in vertex subgroups G_{v_1}, G_{v_2} of G which both contain $\mathbb{Z} \oplus \mathbb{Z}$ as an index 2-subgroup and do not lie in any edge subgroup. Now two cases can occur, according to whether u, v both lie in a same vertex subgroup $v_1 = v_2$ other not. One concludes in each case using the following lemma and a solution to the word problem in G_{v_1} .

Lemma 2.21. *When u, v both lie in vertex subgroup G_{v_1}, G_{v_2} containing $\mathbb{Z} \oplus \mathbb{Z}$ as index 2 subgroups and do not lie in any edge subgroup, then:*

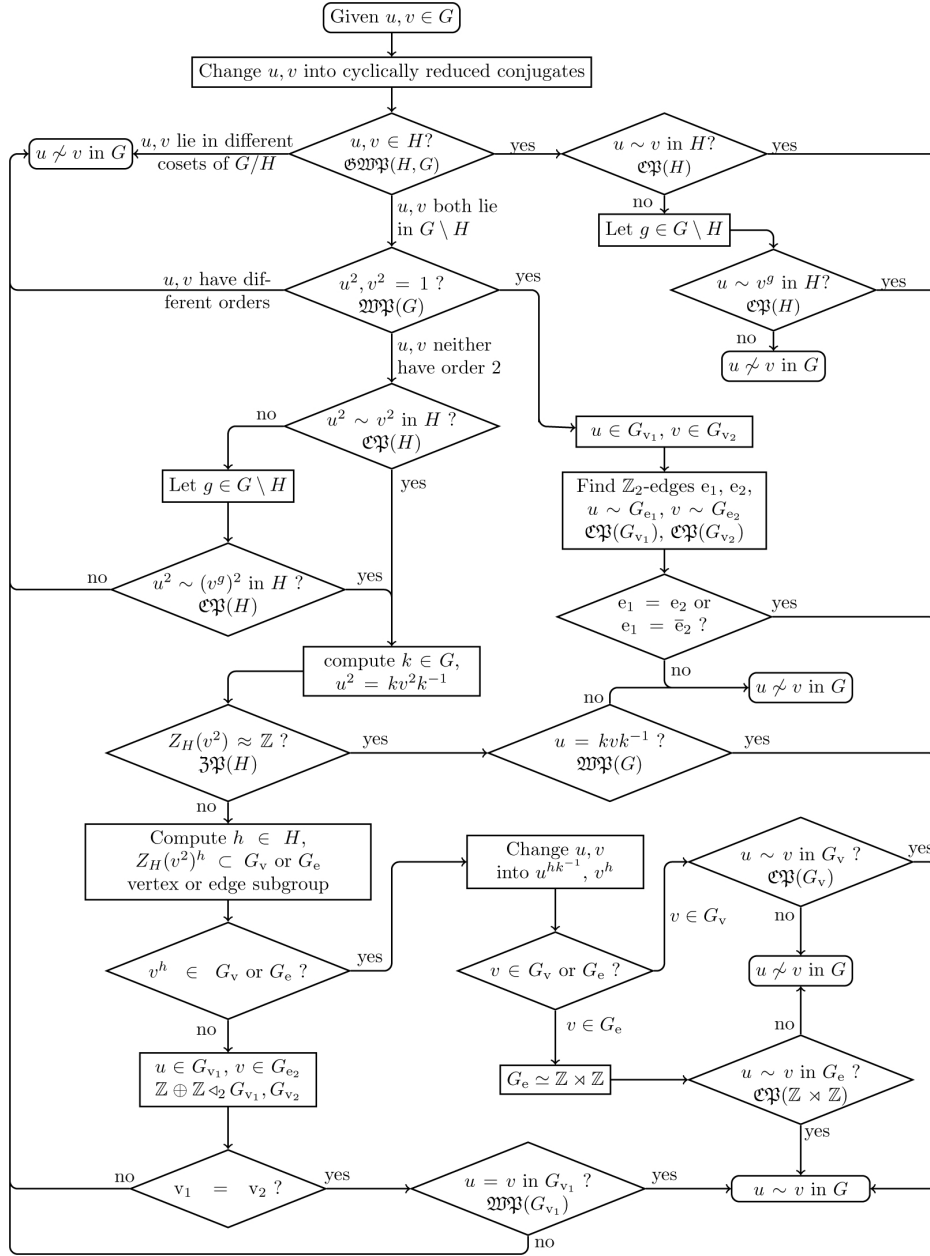


FIGURE 8. The organigram of the algorithm

- (i) if $v_1 \neq v_2$ then u and v are not conjugate in G ,
- (ii) if $v_1 = v_2$, then u and v are conjugate in G if and only if $u = v$.

Proof. Case (2) of Lemma 2.17 cannot occur, since u, v do not lie in an edge subgroup (Lemma 2.16), and neither can occur case (3) since u, v have infinite order: case (1) of Lemma 2.17 occurs.

Denote by G_{e_1}, G_{e_2} the respective $\mathbb{Z} \oplus \mathbb{Z}$ -edge subgroups of G_{v_1}, G_{v_2} (cf. case (1) of Lemma 2.17). Since G_{e_i} is normal in G_{v_i} , $i = 1, 2$, by hypothesis u, v are not conjugate in G_{v_i} to any element in G_{e_i} , $i = 1, 2$.

Case (i). By deleting the pair of edges e_1, \bar{e}_1 in M , G splits as an amalgamated product $K *_{G_{e_1}} G_{v_1}$. The element u lies in the right factor G_{v_1} while v lies in the left factor K . Since u is not conjugate in G_{v_1} to an element of G_{e_1} , it follows from Theorem 4.6 in [MKS] that u and v are not conjugate in G .

Case (ii). Here, $u, v \in G_{v_1}$ and the same argument as in (i) shows that u and v are conjugate in G if and only if they are conjugate in G_{v_1} , and since G_{v_1} is abelian, if and only if they are equal. \square

QED

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